Some applications of random graphs in algebraic and modal logic

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Outline

In 1959, Paul Erdős used probabilistic methods to construct, for each finite k, a finite graph with chromatic number $> k$ and with no cycles of length $< k$.

This theoretical work has practical applications in proving useful negative results in algebraic and modal logic:

A. Str RRA is not elementary,

B. RRA has no canonical axiomatisation (and worse),

C. there are canonical varieties of BAOs that are not elementarily generated.

I will describe these applications.

Relation algebra primer

A *relation algebra* is an algebra $\mathcal{A} = (A, +, -, 0, 1, 1', \breve{}, ;)$ that satisfies certain equations laid down by Tarski in 1941.

RA denotes the variety of relation algebras.

Intended examples: $(\wp(U \times U), \cup, \setminus, \emptyset, U \times U, =, -^{-1}, |\,)$.
RRA denotes the class of subalgebras of products of these.
A relation algebra is said to be *representable* if it is in RRA.

Tarski: RRA is a variety.
Lyndon: RRA $\subset$ RA.
Monk: RRA is not finitely axiomatisable.

Relation algebras are examples of *Boolean algebras with (normal additive) operators* (BAOs).

Duality, atom structures

Consider relation symbols $R_{1'}$ (unary), $R_{\breve{}}$ (binary), $R_{;}$ (ternary).

An *atom structure* is a relational structure of this type.

Given an atom structure $S = (A, R_{1'}, R_{\breve{}}, R_{;})$, we can form its *complex algebra* $\mathrm{Cm}\,S = (\wp(A), \cup, \setminus, \emptyset, A, 1', \breve{}, ;)$, where

- $1' = \{x \in A : S \models R_{1'}(x)\}$,
- $\breve{X} = \{y \in A : \exists x \in X (S \models R_{\breve{}}(x, y))\}$,
- $X\,;Y = \{z \in A : \exists x \in X\ \exists y \in Y (S \models R_{;}(x, y, z))\}$.

Under certain conditions on S, $\mathrm{Cm}\,S$ will be a relation algebra.

We write Str RRA for $\{S : \mathrm{Cm}\,S \in \text{RRA}\}$.

This can all be done for BAOs: we can form $\mathrm{Cm}\,S$; for a variety V of BAOs, $\mathrm{Str}\,\text{V} = \{S : \mathrm{Cm}\,S \in \text{V}\}$.

Graphs

Here, graphs are undirected and loop-free: $G = (V, E)$ where $E \subseteq V \times V$ is irreflexive and symmetric.

For $k \geq 3$, a *cycle of length k* in G is (here) a sequence $v_1, \ldots, v_k \in V$ of distinct nodes with $(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_1) \in E$.

A subset $X \subseteq V$ is *independent* if $E \cap (X \times X) = \emptyset$.

For $k < \omega$, a *k-colouring* of G is a partition of V into k independent sets. The *chromatic number $\chi(G)$* of G is the least $k < \omega$ such that G has a k-colouring, and ∞ if there is no such k.

Example 1 A graph has a 2-colouring iff it has no cycles of odd length.

Relation algebras from graphs

Given a graph $G = (V, E)$, we can form an atom structure $\alpha(G)$:

- The atoms are $1', \mathsf{r}_x, \mathsf{b}_x, \mathsf{g}_x$ (for $x \in V$). Idea: red, blue, green.
- $\alpha(G) \models R_{1'}(a)$ iff $a = 1'$,
- $\alpha(G) \models R_{\smile}(a, b)$ iff $a = b$,
- $\alpha(G) \models R_{;}(a, b, c)$ for all atoms a, b, c except where:
  - one of a, b, c is $1'$ and the other two are distinct,
  - $a = \mathsf{r}_x$, $b = \mathsf{r}_y$, $c = \mathsf{r}_z$, where $\{x, y, z\}$ is independent,
  - similarly for b, g.

We write $\mathcal{A}(G) = \mathrm{Cm}\,\alpha(G)$.

Lemma 2 *For any graph G, $\mathcal{A}(G)$ is a relation algebra.*

Theorem 3 *If G is infinite, then $\mathcal{A}(G) \in$ RRA iff $\chi(G) = \infty$.*

The proof uses games and Ramsey's theorem.

A. Application 1 (joint work with Hirsch)

Question (Maddux, 1982): Is Str RRA an elementary class?

Theorem 4 (Erdős, 1959) *For every finite k, there is a finite graph E_k with chromatic number $> k$ and with no cycles of length $< k$.*

Let $G_k = \bigcup_{l \geq k} E_l$ (disjoint union of graphs), and let $G = \prod_D G_k$ be a non-principal ultraproduct of the G_k. G_k, G are infinite. We have:

- For each k, $\chi(G_k) = \infty$. So $\mathcal{A}(G_k) \in$ RRA, and $\alpha(G_k) \in \mathrm{Str}$ RRA.
- G_k has no cycles of length $< k$. So G has no cycles at all! By example 1, $\chi(G) = 2$. So $\mathcal{A}(G) \notin$ RRA.
- $\mathcal{A}(G) = \mathrm{Cm}\,\alpha(\prod_D G_k) \cong \mathrm{Cm} \prod_D \alpha(G_k)$ (easy to check). So $\prod_D \alpha(G_k) \notin \mathrm{Str}$ RRA.

Hence, Str RRA is not closed under ultraproducts and so is not elementary. *Answer to Maddux: 'no'.*

B. Canonicity

For a relation algebra or BAO $\mathcal{A}$, the set of ultrafilters of $\mathcal{A}$ forms an atom structure $\mathcal{A}_+$. The *canonical extension* of $\mathcal{A}$ is $\mathcal{A}^\sigma = \mathrm{Cm}(\mathcal{A}_+)$.

- A class $\mathcal{K}$ of relation algebras/BAOs is *canonical* if $\mathcal{A} \in \mathcal{K} \Rightarrow \mathcal{A}^\sigma \in \mathcal{K}$.
- An axiom τ is *canonical* if $\mathcal{A} \models \tau \Rightarrow \mathcal{A}^\sigma \models \tau$.

Monk showed that RRA is canonical.

Question (Venema, ∼1995): Does RRA have a canonical axiomatisation – each individual axiom in it is canonical?

Preliminary 1: 'local' version of theorem 3

Theorem 3 said that for infinite G, $\mathcal{A}(G)$ is representable iff $\chi(G) = \infty$.

In fact, the higher $\chi(G)$ is, the nearer $\mathcal{A}(G)$ gets to being representable:

Proposition 5 *Let Σ be any axiomatisation of* RRA. *Then*

1. *For all $k < \omega$, there is finite $X \subseteq \Sigma$ such that for all infinite G, if $\mathcal{A}(G) \models X$ then $\chi(G) > k$.*
2. *For any finite $X \subseteq \Sigma$, there is $m < \omega$ such that for all G, if $\chi(G) \geq m$ then $\mathcal{A}(G) \models X$.*

This can be proved by games, Ramsey arguments, and first-order compactness.

Preliminary 2: Erdős++

Using a variant of Erdős's argument, it can be shown that

Proposition 6 *For any $2 \leq m < \omega$, there is an inverse system*

$$\mathcal{S}: \quad G_0 \overset{\pi_0}{\longleftarrow} G_1 \overset{\pi_1}{\longleftarrow} \cdots$$

of finite graphs G_i with $\chi(G_i) = m$, and surjective 'p-morphisms' π_i, such that $G = \lim_{\leftarrow} \mathcal{S}$ is infinite and $\chi(G) = 2$.

p-morphism duality gives a direct system $\mathcal{A}(G_0) \to \mathcal{A}(G_1) \to \cdots$ of finite relation algebras and embeddings. Let $\mathcal{D} = \lim_{\to} \mathcal{A}(G_i)$.

Lemma 7 (essentially Goldblatt) $\mathcal{D}^\sigma \cong \mathcal{A}(G)$.

Application 2 (joint work with Venema)

Theorem 8 *There is no canonical equational axiomatisation of* RRA.

Proof. If there were such an axiomatisation Σ, proposition 5 gives

1. finite $X \subseteq \Sigma$ such that for any infinite G, $\mathcal{A}(G) \models X \Rightarrow \chi(G) > 2$,
2. $m < \omega$ such that if $\chi(G) \geq m$ then $\mathcal{A}(G) \models X$.

Proposition 6 gives an inverse system $G_0 \leftarrow G_1 \leftarrow \cdots$ with $\chi(G_i) = m$ (all i), and infinite inverse limit G with $\chi(G) = 2$.

$\chi(G_i) = m$, so $\mathcal{A}(G_i) \models X$ for all i. Hence $\mathcal{D} = \lim_{\to} \mathcal{A}(G_i) \models X$.

X is canonical, so $\mathcal{D}^\sigma \models X$.

But $\mathcal{D}^\sigma \cong \mathcal{A}(G)$, so $\mathcal{A}(G) \models X$; and G is infinite. Hence $\chi(G) > 2$, contradiction. ■

Worse: can strengthen to show RRA has no axiomatisation with only finitely many non-canonical first-order sentences.

C. Canonicity and elementary generation

A variety V of BAOs is *elementarily generated* if it is generated by $\{\mathrm{Cm}\, S : S \in \mathcal{K}\}$ for some elementary class $\mathcal{K}$.

Theorem 9 (mostly Goldblatt 1989, extending Fine 1975)

1. *Any elementarily generated variety is canonical.*
2. *Write $\mathrm{Cst}\, \mathrm{V} = \{\mathcal{A}_+ : \mathcal{A} \in \mathrm{V}\}$. Recall $\mathrm{Str}\, \mathrm{V} = \{S : \mathrm{Cm}\, S \in \mathrm{V}\}$.*
   - *V is canonical iff $\mathrm{Cst}\, \mathrm{V} \subseteq \mathrm{Str}\, \mathrm{V}$.*
   - *V is elementarily generated iff any ultraproduct of structures in $\mathrm{Cst}\, \mathrm{V}$ is in $\mathrm{Str}\, \mathrm{V}$.*

Question (Fine 1975, Goldblatt): does the converse of (1) hold?

Chromatic number generalised to algebras

We can regard a graph $G = (V, E)$ as an atom structure of the BAO $\mathrm{Cm}\, G = (A, +, -, 0, 1, e, d)$, where $A = \wp(V)$ and for $a \in A$,

- $e(a) = \{x \in V : \exists y \in a((x, y) \in E)\}$ (neighbours of nodes in a),
- $d(a) = \begin{cases} 1, & \text{if } a > 0, \\ 0, & \text{otherwise.} \end{cases}$

For a BAO $\mathcal{A}$ of this type, and $k < \omega$, say $\chi(\mathcal{A}) \leq k$ if

$$\mathcal{A} \models \exists x_0 \ldots x_{k-1} \Big(\sum_{i<k} x_i = 1 \wedge \bigwedge_{i<k} (x_i \cdot e(x_i) = 0) \Big).$$

$\chi(\mathcal{A})$ is the least k with $\chi(\mathcal{A}) \leq k$, if any; ∞ if not.

Note: $\chi(\mathrm{Cm}\, G) = \chi(G)$ for any graph G.

Application 3 (joint work with Goldblatt, Venema)

Recall E_k is a finite graph with $\chi(E_k) > k$ and with no cycles of length $< k$. We can assume $|E_0| < |E_1| < \cdots$.

Let $\mathcal{K} = \{\mathcal{A} : \forall k < \omega(|\mathcal{A}| \geq 2^{|E_k|} \Rightarrow \chi(\mathcal{A}) > k)\}$ — elementary. Let V be the variety generated by $\mathcal{K}$.

Lemma 10 V *is canonical.*

Lemma 11 *For each* k, $\mathrm{Cm}\, E_k \in \mathcal{K} \subseteq$ V*.*
As $\mathrm{Cm}\, E_k$ *is finite, we have* $E_k \cong (\mathrm{Cm}\, E_k)_+ \in \mathrm{Cst}\,$V*.*

Let D be a non-principal ultrafilter on ω. As usual, $E = \prod_D E_k$ has no cycles, so $\chi(E) = 2$.

Lemma 12 $E \notin \mathrm{Str}\,$V.

Conclude by Goldblatt's test that V *is a canonical variety that is not elementarily generated.*

Conclusion

We saw three applications of Erdős graphs:

A. $\mathrm{Str}\,$RRA is not elementary,

B. RRA has no canonical axiomatisation (and worse),

C. there are canonical varieties of BAOs that are not elementarily generated.

Le Bars used random graphs to show failure of 0–1 law for frame satisfiability in propositional modal logic.

Probabilistic constructions are very powerful. Perhaps there will be other applications of them in AL/ML. . .