Undecidability of various problems of representing binary relations

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RRA is the class of *representable relation algebras* — the closure under isomorphism of the class of fields of binary relations equipped with the boolean operators, identity ($1'$), converse ($\smile$) and composition (;) operators. When Tarski formulated the axioms for Relation Algebra it seems he rather hoped that his axioms would provide a complete axiomatisation of the validities **RRA**. If this had been the case then the algebraic approach to reasoning about relations would have had an enormous advantage over the more commonly used predicate logic. However, Tarski's axioms are not complete [Lyn50] and no finite set of formulas can be sound and complete over **RRA** [Mon64]. Since then the list of negative results about relation algebra has grown considerably and we will outline some of these results here. For more detailed surveys see [Sch91, HH02, Mad06].

Any sound and complete equational axiomatisation of **RRA** must involve infinitely many variables [Jón91], it cannot consist only of Sahlqvist equations [Ven97], indeed it must involve infinitely many non-canonical equations [HV05]. Tarski showed that the equational theory of relation algebras, although by definition finitely axiomatisable, is not decidable. He also showed that the equational theory of **RRA** is not decidable [TG87, theorem 8.5(xii)(β)]. In [AGN94] a reduction of the word problem for semigroups to relation algebra is used to show that various important subclasses of relation algebras are undecidable.

Hirsch and Hodkinson used the undecidability of the tiling problem to prove that the class of finite relation representable relation algebras is not recursive [HH01]. The undecidability of the equational theory of representable relation algebras is a corollary of this. One problem that was not resolved, though, was the decidability of the class of finite relation algebras with a representation on a finite base. This problem remains open and deserves attention.

Given all these negative results for classical representations of relation algebras two other approaches were considered. One was to consider non-classical representations, in particular *relativized representations* and here the results are encouraging. Every relation algebra (indeed, every *weakly associative algebra*) has a representation relativized to some reflexive, symmetric relation [Mad82], and the class of relation-type algebras with unrestricted relativized representations is finitely axiomatizable [Mar99]. Every finite relation algebra has a relativized representation on a finite base [AHN99] and the equational theory of relativized relation algebras is decidable.

Item	Signature S	Fin. ax.	Dec. Eq. Th.	Dec. Rep. Fin. Alg.
1	$S = \mathbf{RA}$	$\times$[Mon64]	$\times$[TG87]	$\times$[HH01]
2	$S = \mathbf{BA}$	$\surd$[Sto36]	$\surd$	$\surd$[Sto36]
3	$S \subseteq \{0, 1, +, -, \leq, 1', \smile\}$	$\surd$[Sch91]	$\surd$[Ném87]	$\surd$
4	$S \subseteq \{0, 1, +, ., \leq, 1', \smile, ;\}$	d	$\surd$[AB95]	d
5	$S \supseteq \{., \smile, ;\}$	$\times$[HM00]	d	?
6	$S \supseteq \{+, ., ;\}$	$\times$[And91]	d	?
7	$S \supseteq \{+, ., 1', ;\}$	$\times$	d	$\times$Follows from [HH01]
8	$S = \{+, ;\}$	$\times$[And89]	$\surd$	?
9	$S = \{., 1', ;\}$	$\times$[HM07]	$\surd$	?
10	$S = \{\leq, ;\}$	$\surd$[Sch91]	$\surd$	$\surd$[Sch91]
11	$S = \{\leq, 1', ;\}$	$\times$[Hir05]	$\surd$	?

Figure 1: Finite axiomatisability of $\mathbf{R}(S)$, decidability of the equational theory of $\mathbf{R}(S)$ and decidability of representability for finite S-algebras, for various subsignatures S of the signature of relation algebra. $\surd$ means "yes", $\times$ means "no" and "d" means that the result depends on the choice of S.

Another important approach has been to consider alternative sets of operators for binary relations. Consider a signature S consisting of some of the operators definable by the relation algebra operators, $S \subseteq \{0, 1, ., +, -, \leq, 1', \smile, ;\}$ where $\leq$ is the binary relation which must be interpretted as set inclusion in a representation (if $+$ or . are included in S they must be interpretted as union, intersection of sets respectively, so $\leq$ is automatically interpretted as inclusion). We can weaken the notion of a representability so that a representation is only required to respect the operators included in S. We write $\mathbf{R}(S)$ to denote the class of all S-algebras isomorphic to fields of binary relations equipped with the operators from S. Figure 1 summarizes some of the main results. The undecidability of the representation problem for finite relation algebras result, mentioned above, can be extended to show that representability is not decidable for finite algebras for any subsignature of relation algebra containing $\{., +, 1', ;\}$. Also note that the equational theory of any positive relation algebra sub-signature is decidable [AB95]. A corollary is that if $\{., +, 1', ;\} \subseteq S \subseteq \{0, 1, ., +, 1', \smile, ;\}$ then $\mathbf{R}(S)$ is not a variety.

Having surveyed some of the previously known undecidability results we will now introduce some new results recently obtained. A *partial algebra* is a set together with a partial binary operation.

THEOREM 1 ([Eva53, JV09]) *The following classes are not recursive.*

1. *The class of finite partial algebras that embed into a group.*
2. *The class of finite partial algebras that embed into a finite group.*

Based on the previous theorem we can derive the following new results.

THEOREM 2 *The following classes are not recursive.*

1. *The class of finite boolean monoids that embed into (finite) relation algebras.*

2. *The class of finite representable boolean monoids.*

3. *The class of finite boolean monoids representable over a finite set.*

In this talk I will outline reductions from the partial algebra embeddability problems of theorem 1 to those of theorem 2 and thereby establish these new results.

PROBLEM 3 *Is the set of finite sequential algebras that embed into a (finite) relation algebra recursive?*

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