Weakly associative relation algebras with projections

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Built on the foundations laid by Peirce, Schröder and others in the 19th century, the modern development of relation algebras started with the work of Tarski and his colleagues [21, 22]. They showed that relation algebras can capture strong first-order theories like *ZFC*, and so their equational theory is undecidable. The less expressive class WA of weakly associative relation algebras were introduced by Maddux [7]. Németi [16] showed that WAs have a decidable universal theory. There has been extensive research on increasing the expressive power of WA by adding new operations [1, 4, 11, 13, 20]. Extensions of this kind usually also have decidable universal theories. Here we give an example—extending WAs with set-theoretic projection elements—where this is not the case. These 'logical' connectives are set-theoretic counterparts of the axiomatic quasi-projections that have been investigated in the representation theory of relation algebras [6, 19, 22]. We prove that the quasi-equational theory of the extended class PWA is not recursively enumerable. By adding the difference operator D one can turn WA and PWA to discriminator classes where each universal formula is equivalent to some equation. Hence our result implies that the projections turn the decidable equational theory of '$\mathsf{WA} + D$' to non-recursively enumerable.

1 Introduction and results

A *weakly associative relation algebra* (WA) is an algebra of the form

$$\mathfrak{A} = \langle A, W, \cap, -^W, \mid^W, ^{-1}, Id_U \rangle,$$

where A (the *universe* of $\mathfrak{A}$) is a nonempty collection of binary relations between elements of some set U (the *base* of $\mathfrak{A}$) such that $\langle A, W, \cap, -^W \rangle$ is a Boolean set algebra with its *unit* W being a reflexive and symmetric binary relation on U, and

$$Id_U \overset{\text{def}}{=} \{\langle u, u \rangle : u \in U\},$$

and for all $R, S \in A$,

$$R \mid^W S \overset{\text{def}}{=} \{\langle u, v \rangle \in W : \exists w\, (\langle u, w \rangle \in R \text{ and } \langle w, v \rangle \in S)\},$$
$$R^{-1} \overset{\text{def}}{=} \{\langle u, v \rangle : \langle v, u \rangle \in R\}$$

belong to A. We describe properties of WAs in the first-order language (with equality) of the similarity type $t^{\,ra}$ that consists of the Booleans $1, \cdot, -$, binary function symbol $;$ (for composition), unary function symbol $\breve{}$ (for converse) and constant $1'$ (for identity). (We will also use 0 and binary $-$ in the usual sense, and $x \leq y$ as a shorthand for $x \cdot y = x$.)

WAs were introduced by Maddux [7], who showed that they can be axiomatised by finitely many $t^{\,ra}$-type equations. The name 'weakly associative' comes from the axiom

$$\text{(wa)} \quad (1' \cdot x)\,;(1\,;1) = ((1' \cdot x)\,;1)\,;1,$$

which is a weaker form of associativity of relation composition.

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If the unit element W of a WA is the full Cartesian product $U \times U$ of its base U then the algebra is called a *set relation algebra*. Composition in set relation algebras is fully associative and computationally they are more complex than WAs: While even the universal theory of WAs is decidable [16], set relation algebras have an undecidable (though recursively enumerable) equational theory [3,21,22].

Tarski [21,22] introduced the so-called *quasi-projections* in an attempt to characterise set relation algebras by t^{ra}-type equations. Quasi-projections are elements p, q of a t^{ra}-type algebra that satisfy the following equations:

$$\text{(qp1)} \qquad 1 \leq p^{\smile} \,;\, q$$

$$\text{(qp2)} \qquad p^{\smile} \,;\, p \leq 1'$$

$$\text{(qp3)} \qquad q^{\smile} \,;\, q \leq 1'.$$

Take any nonempty set U_0 that does not contain ordered pairs, and let U be the *closure* of U_0 under forming ordered pairs, that is,

$$U \stackrel{\text{def}}{=} \bigcup_{n\in\omega} U_n, \quad \text{where} \quad U_{n+1} \stackrel{\text{def}}{=} \Big(\bigcup_{i\leq n} U_i\Big) \times \Big(\bigcup_{i\leq n} U_i\Big). \tag{1}$$

The standard examples of quasi-projections are the ‘real’ set-theoretic projection functions (taken as binary relations) over such a set U:

$$P_U \stackrel{\text{def}}{=} \{\langle\langle u,x\rangle,u\rangle \,:\, u\in U,\ \langle u,x\rangle \in U\} \qquad Q_U \stackrel{\text{def}}{=} \{\langle\langle x,u\rangle,u\rangle \,:\, u\in U,\ \langle x,u\rangle \in U\}.$$

Observe that if P_U and Q_U are elements of a set relation algebra with base U then (qp1)–(qp3) hold in the algebra: (qp1) expresses that $\langle u,v\rangle \in U$ whenever $u,v \in U$, while (qp2) and (qp3) simply say that the projections are functions.

Maddux [8,10] added the projections to the similarity type, that is, considered the extension t^{ra}_{pq} of t^{ra} with two constant symbols p and q. He introduced *true pairing algebras* as t^{ra}_{pq}-type algebras of the form

$$\langle \mathfrak{Ra}(U), P_U, Q_U\rangle,$$

where U is a set as in (1) and $\mathfrak{Ra}(U)$ is the set relation algebra of *all* subsets of $U \times U$ (see also [5]). Sain and Simon [14,18] showed that the t^{ra}_{pq}-type equational theory of true pairing algebras becomes very complex, in fact it is Π^1_1-complete.

There has been extensive research on increasing the expressive power of WAs by adding new operations [1,4,11,13,20]. Extensions of this kind usually also turned out to have decidable universal theories. In this paper we discuss what happens if we add set-theoretic projections to WAs. As P_U and Q_U are not necessarily subsets of the unit W of a WA with base U, the sensible operations to add are their ‘W-relativised’ versions, just like in the definition of composition in WAs. This way we define a PWA to be a t^{ra}_{pq}-type algebra

$$\mathfrak{A} = \langle A, W, \cap, -^W, \,|^W, {}^{-1}, Id_U, P_U \cap W, Q_U \cap W\rangle$$

such that $\langle A, W, \cap, -^W, \,|^W, {}^{-1}, Id_U\rangle$ is a WA. Observe that, while (qp2) and (qp3) hold in every PWA, (qp1) might not. We call those PWAs where (qp1) hold *closed*. (Note that the base U of a closed PWA is not necessarily closed under forming ordered pairs: If a pair $\langle u,v\rangle$ does not belong to the unit, then it can happen that u and v are in U but $\langle u,v\rangle$ is not.)

Let $\mathfrak{Tpa}_u$ denote the true pairing algebra whose base U is the closure of the singleton set $U_0 = \{u\}$ for some non-pair set u (see (1)). Clearly, $\mathfrak{Tpa}_u$ is a PWA. Recall that a *quasi-equation* is a formula of the form $\varphi(\bar{x}) \to \psi(\bar{x})$, where $\varphi(\bar{x})$ is a conjunction of equations and $\psi(\bar{x})$ is an equation. Our main result is the following:

Theorem 1.1 *Let* K *be a class of* PWA*s such that* $\mathfrak{Tpa}_u$ *belongs to* K *for some non-pair set* u. *Then the quasi-equational theory of* K *is not recursively enumerable. In particular, the quasi-equational theories of all* PWA*s and all closed* PWA*s are not recursively enumerable.*

WAs do not form a discriminator class (see e.g. [9]), that is, not every universal formula is equivalent to an equation over WAs. A similar argument shows that PWAs do not form a discriminator class either, thus Theorem 1.1 does not say anything about their equational theory. However, one can add a discriminator term to WAs and PWAs e.g. by introducing the following *difference operator* D. Let $\mathfrak{A}$ be a WA or PWA. For every X in $\mathfrak{A}$, let

$$D^{\mathfrak{A}}(X) \stackrel{\text{def}}{=} \begin{cases} 1^{\mathfrak{A}} & \text{if } |X| > 1, \\ 1^{\mathfrak{A}} - X & \text{if } |X| = 1, \\ \emptyset & \text{else.} \end{cases}$$

(Observe that in set relation algebras the t^{ra}-type term $-\big(-(1 \mathbin{;} x \mathbin{;} -1') \cdot -(-1' \mathbin{;} x \mathbin{;} 1)\big)$ defines such a D.) We denote the resulting algebra by $\mathfrak{A}^D$, and call it a WAD or a PWAD, respectively. WADs were introduced and studied by Mikulás [13], who showed that their equational theory is decidable. As a consequence of Theorem 1.1 we obtain that the projections 'ruin' the computational behaviour of WADs:

Theorem 1.2 *Let* K *be a class of* PWAD*s such that* $\mathfrak{Tpa}_u^D$ *belongs to* K *for some non-pair set* u*. Then the equational theory of* K *is not recursively enumerable. In particular, the equational theories of all* PWAD*s and all closed* PWAD*s are not recursively enumerable.*

We prove Theorems 1.1 and 1.2 by giving a reduction of the well-known [2,12] non-recursively enumerable set of unsolvable Diophantine equations. Throughout, t_ω denotes the similarity type of arithmetic, that is, t_ω consists of a constant 0, unary function symbol $succ$ and binary function symbols $+$ and $*$. By a *Diophantine equation* we mean a t_ω-type atomic formula. We let (with a slight abuse of notation)

$$\underline{\omega} \stackrel{\text{def}}{=} \langle \omega, 0, succ, +, * \rangle$$

denote the *standard model of arithmetic*.

We will give a recursive translation of Diophantine equations to t^{ra}_{pq}-type quasi-equations such that for any Diophantine equation φ, φ is unsolvable in $\underline{\omega}$ iff its translation q_φ is valid in K. The quasi-equation q_φ is of the form $\Psi \to (\tau_\varphi = 0)$, where Ψ is a conjunction of equations that 'forces' some part of PWAs to 'behave like' $\underline{\omega}$, and the translation of φ into the t^{ra}_{pq}-type term τ_φ 'preserves the behaviour of numbers.'

The idea of using unsolvable Diophantine equations as the master problem for complexity issues in algebras of relations comes from Németi [17]. Our reduction is a refinement of the one used by Sain and Simon [18] for defining arithmetic in true pairing algebras. Like [18], first we also define numbers and the standard model of arithmetic in our algebras (Sections 2–4), and then we turn Diophantine equations to formulas of a relational similarity type containing binary predicates only. Then we use the method of Tarski and Givant [22] to translate these 'all-binary' formulas to relation algebraic terms (Section 5).

Each of these steps has been designed to work having true pairing algebras in mind, that is, when all the necessary pairs are present in the unit. In general, this is not the case for 'relativised' relation algebras like PWAs. We are dealing with the issue of forcing the existence of necessary pairs in Section 3. Another problem that had to be solved was that the lack of a discriminator term made it impossible to define numbers in PWAs uniquely, and we had to deal with a whole set of elements 'mimicking' each number instead. Finally, in Section 5 we discuss why the Tarski-Givant translation works in PWAs for the restricted kind of all-binary formulas obtained from Diophantine equations.

None of these tricks helped to overcome the final hurdle: we needed a discriminator term to convert our quasi-equation to an equation. So the the following question remains open:

Problem. Is the equational theory of PWAs decidable?

2 Defining numbers in PWAs

To begin with, let us recall some basic computations in WAs. Consider the following unary t^{ra}-type terms:

$$\begin{aligned} \mathsf{Do}\, x &\stackrel{\text{def}}{=} (x \mathbin{;} 1) \cdot 1' \\ c_0 x &\stackrel{\text{def}}{=} 1 \mathbin{;} \big((1 \mathbin{;} x) \cdot 1'\big) \\ c_1 x &\stackrel{\text{def}}{=} \big((x \mathbin{;} 1) \cdot 1'\big) \mathbin{;} 1 \,. \end{aligned}$$

For any binary relation X, we let $dom\, X$ denote the *domain* of X. A routine computation shows that, for every WA $\mathfrak{A}$ and every X in $\mathfrak{A}$,

$$
\begin{aligned}
(\mathsf{Do}\, x)^{\mathfrak{A}}[x/X] &= \{\langle u,u\rangle : u \in dom\, X\} \\
(c_0 x)^{\mathfrak{A}}[x/X] &= \{\langle u,v\rangle \in 1^{\mathfrak{A}} : \exists z\ \langle z,v\rangle \in X\} \\
(c_1 x)^{\mathfrak{A}}[x/X] &= \{\langle u,v\rangle \in 1^{\mathfrak{A}} : \exists z\ \langle u,z\rangle \in X\}\,.
\end{aligned}
$$

(Note that in set relation algebras one can use the simpler terms $1\ ;\ x$ and $x\ ;\ 1$ for $c_0 x$ and $c_1 x$, respectively, but they might not define the above sets in an arbitrary WA.)

For any set u, we can define 'numbers' by taking $u^{(0)} \stackrel{\text{def}}{=} \langle u,u\rangle$, $u^{(k+1)} \stackrel{\text{def}}{=} \langle u^{(k)}, u^{(k)}\rangle$. These numbers belong to the base U of the true pairing algebra $\mathfrak{Tpa}_u$. Moreover, their collection

$$N_u \stackrel{\text{def}}{=} \{u^{(k)} : k \in \omega\}$$

is an element of $\mathfrak{Tpa}_u$, with $(P_U \cap Q_U)^{-1}$ acting as successor function on it.

Below we discuss how to define numbers and the successor function in any PWA. To this end, take some variable ν (intended to represent the set ω of numbers), and define a t^{ra}_{pq}-type term ζ by taking

$$\zeta \stackrel{\text{def}}{=} \nu - (\mathsf{Do}\, p \cdot \mathsf{Do}\, q)\,.$$

Observe that ζ defines the number 0 in $\mathfrak{Tpa}_u$ in the sense that $\zeta^{\mathfrak{Tpa}_u}[\nu/N_u] = \{u^{(0)}\}$.

From now on, we write $x\,;y\,;z$ without parentheses, whenever any of x, y, z is below $1'$ (since weak associativity (wa) holds in PWAs). Let $\Phi^1_\omega(\nu)$ be the conjunction of the following t^{ra}_{pq}-type equations:

(e1) $\nu \le 1'$

(e2) $\nu \le 1\,;\nu\,;(p\cdot q)$

(e3) $1 \le 1\,;\zeta\,;1$

(e4) $\nu\,;p = \nu\,;q$

(e5) $\nu\,;p \le 1\,;\nu$

It is straightforward to check the following:

Lemma 2.1 $\mathfrak{Tpa}_u \models \Phi^1_\omega[\nu/N_u]$.

On the other hand, we show that these equations force ν to simulate natural numbers in any PWA (see Fig. 1 for the idea). To this end, we fix some PWA $\mathfrak{A}$ (with base U) and an element N in $\mathfrak{A}$.

Lemma 2.2 *If* $\mathfrak{A} \models \Phi^1_\omega[\nu/N]$ *then the following hold:*

(i) $N \subseteq Id_U$.

(ii) *For every* $\langle u,u\rangle \in N$, $\big\langle\langle u,u\rangle, \langle u,u\rangle\big\rangle \in N$.

(iii) *The set* $Z_N \stackrel{\text{def}}{=} \{u \in U : \langle u,u\rangle \in N \text{ and } u \notin dom\ p^{\mathfrak{A}} \cap dom\ q^{\mathfrak{A}}\}$ *is not empty.*

(iv) *For every* $\langle u,u\rangle \in N$, *if* $u \notin Z_N$ *then* $u \in N$.

(v) *For every* $u \in Z_N$,

- $succ_u \stackrel{\text{def}}{=} (p^{\mathfrak{A}} \cap q^{\mathfrak{A}})^{-1} \cap (N_u \times N_u)$ *is a total function on* N_u,
- $\langle N_u, \langle u,u\rangle, succ_u\rangle$ *is isomorphic to* $\langle \omega, 0, succ\rangle$.

(vi) $N = \bigcup_{u \in Z_N} N_u$.

P r o o f. Items (i) and (ii) hold by (e1) and (e2), respectively. As the term ζ defines the set $\{\langle u,u\rangle : u \in Z_N\}$ in $\mathfrak{A}$, (iii) holds by (e3).

For (iv): If $\langle u,u\rangle \in N$ and $u \notin Z_N$ then $u \in dom\ p^{\mathfrak{A}} \cap dom\ q^{\mathfrak{A}}$, so $u = \langle x,y\rangle$ for some x,y, and we have $\big\langle\langle x,y\rangle,x\big\rangle \in p^{\mathfrak{A}}$ and $\big\langle\langle x,y\rangle,y\big\rangle \in q^{\mathfrak{A}}$. Therefore, $x=y$ by (e1) and (e4). Now $u=\langle x,x\rangle \in N$ follows by (e1) and (e5).

Item (v) is obvious from the definitions of N_u and $succ_u$.

For (vi): First, $N_u \subseteq N$ by (ii), for all $u \in Z_N$. Now assume that $\langle x,x\rangle \in N$, but $\langle x,x\rangle \neq u^{(k)}$ for any $u \in Z_N$, $k \in \omega$. Then, by (iv) and (i), for every $k \in \omega$ we can define some $x_k \in N$ by taking $x_0 \stackrel{\text{def}}{=} x$ and $x_{k+1} \stackrel{\text{def}}{=} (p^{\mathfrak{A}} \cap q^{\mathfrak{A}})(x_k)$. So we have $x_k = \langle x_{k+1},x_{k+1}\rangle = \big\{\{x_{k+1}\}\big\}$ for every $k \in \omega$, and so

$$x_0 \ni \{x_1\} \ni x_1 \ni \{x_2\} \ni x_2 \ni \ldots$$

is an infinite descending $\in$-chain, which cannot exist in *ZFC*. □

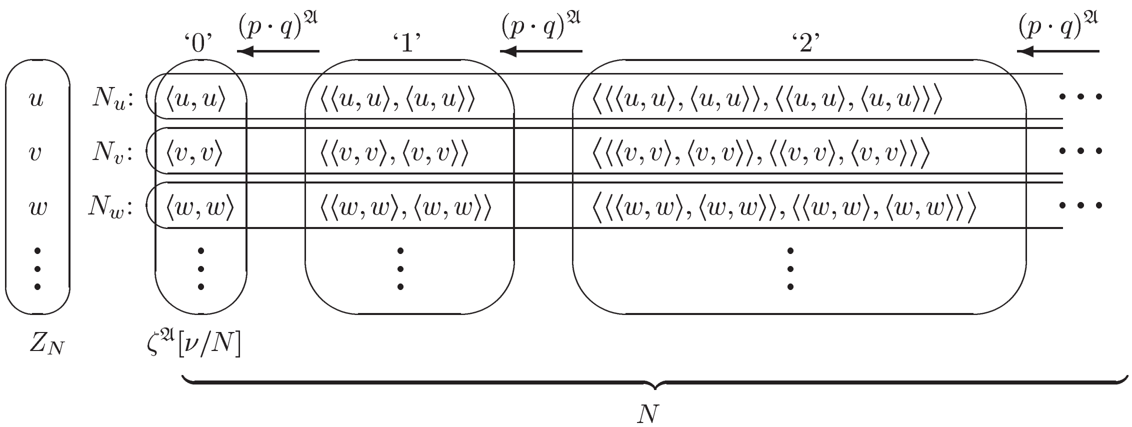

Fig. 1 Numbers in a PWA.

3 Defining sequences of numbers

We need to have not only the successor function, but addition and multiplication as well, so we have to deal with pairs and triples of numbers. Moreover, in order to encode some of their properties, we need to be able to handle sequences of even longer length.

None of these is a problem in $\mathfrak{Tpa}_u$. As its base U is closed under forming ordered pairs and its unit is $U \times U$, all these sequences are in its unit. However, in an arbitrary PWA we have to 'force' them to be present, as we will see below. Another problem is that, as we have seen in the previous section, we may have multiple 'zeros', that is, the set Z_N can have many elements. As we intend to define addition and multiplication in the usual recursive way, we have to exclude somehow 'mixed' sequences, like $\big\langle\langle v^{(5)},w^{(3)}\rangle,u^{(2)}\big\rangle$ for distinct zeros u,v,w. In other words, we want a number-tuple like $\big\langle\langle 5,3\rangle,2\big\rangle$ to be represented by the set $\big\{\big\langle\langle u^{(5)},u^{(3)}\rangle,u^{(2)}\big\rangle : u \in Z_N\big\}$.

We begin with defining the *powers* of a set N. Let $Pow_0(N) \stackrel{\text{def}}{=} N$ and for any $k \in \omega, k>0$, let

$$Pow_k(N) \stackrel{\text{def}}{=} \{K \times L \ :\ K \in Pow_i(N),\ L \in Pow_j(N),\ i,j<k\},$$

and let $Pow(N) \stackrel{\text{def}}{=} \bigcup_{k\in\omega} Pow_k(N)$. For any $K \in Pow(N)$, we let

$$rk(K) \stackrel{\text{def}}{=} min\{k \in \omega : K \in Pow_k(N)\}\,,$$

that is, $rk(K)$ is the number of 'N-components' in K. For any $K,L \in Pow(N)$, we define

$$K \sqsubset L \qquad \text{iff} \qquad rk(K) < rk(L)\,.$$

Clearly, $\sqsubset$ is a strict partial order.

As we mentioned above, even if $N \subseteq 1^{\mathfrak{A}}$, in general K is not necessarily a subset of $1^{\mathfrak{A}}$, for every $K \in Pow(N)$. We let $K|_{1^{\mathfrak{A}}}$ denote the set of all elements from K that 'hereditarily' belong to $1^{\mathfrak{A}}$, that is $N|_{1^{\mathfrak{A}}} \stackrel{\text{def}}{=} N$ and $(K_1 \times K_2)|_{1^{\mathfrak{A}}} \stackrel{\text{def}}{=} (K_1|_{1^{\mathfrak{A}}} \times K_2|_{1^{\mathfrak{A}}}) \cap 1^{\mathfrak{A}}$.

We show that $K|_{1^{\mathfrak{A}}}$ is definable in terms of N in any PWA. To this end, let $\sigma_N \stackrel{\text{def}}{=} \nu$, and for any $K \in Pow(N)$, $K = K_1 \times K_2$, let

$$\sigma_K \stackrel{\text{def}}{=} c_1\,[(p\,;\sigma_{K_1}) \cdot q] \cdot c_0\,[(\sigma_{K_2}^{\smile}\,;p^{\smile}) \cdot q^{\smile}]\,. \tag{2}$$

Further, for any $K \in Pow(N)$, we let

$$\mathsf{id}_K \stackrel{\text{def}}{=} c_1[(p\,;\sigma_K) \cdot q] \cdot 1'. \tag{3}$$

Now recall the equation (qp1) (expressing closedness of a PWA). Observe that if (qp1) holds in a PWA $\mathfrak{A}$ then $\langle\langle a,b\rangle,a\rangle \in p^{\mathfrak{A}}$, $\langle\langle a,b\rangle,b\rangle \in q^{\mathfrak{A}}$, and $\langle\langle a,b\rangle,\langle a,b\rangle\rangle \in 1'^{\mathfrak{A}}$, whenever $\langle a,b\rangle \in 1^{\mathfrak{A}}$.

The following lemma shows the meaning of the terms σ_K and id_K in closed PWAs:

Lemma 3.1 *If* $\mathfrak{A} \models \text{(qp1)} \wedge \Phi^1_\omega[\nu/N]$ *then, for every* $K \in Pow(N)$, *we have*

(i) $K|_{1^{\mathfrak{A}}} \;=\; \sigma_K^{\mathfrak{A}}[\nu/N]$, *and*

(ii) $\{\langle x,x\rangle : x \in K|_{1^{\mathfrak{A}}}\} \;=\; \mathsf{id}_K^{\mathfrak{A}}[\nu/N]$.

P r o o f. For (i): We prove the statement by induction on the structure of K. The case of $K = N$ is obvious. Assume now that $K = K_1 \times K_2$. First, take some $x = \langle u,v\rangle \in K|_{1^{\mathfrak{A}}}$, $u = \langle a,b\rangle \in K_1|_{1^{\mathfrak{A}}}$, $v = \langle c,d\rangle \in K_2|_{1^{\mathfrak{A}}}$. By (qp1), $\big\langle\langle a,b\rangle,a\big\rangle \in p^{\mathfrak{A}}$, $\big\langle\langle a,b\rangle,b\big\rangle \in q^{\mathfrak{A}}$. Therefore, by the induction hypothesis, $\big\langle\langle a,b\rangle,b\big\rangle \in ((p\,;\sigma_{K_1}) \cdot q)^{\mathfrak{A}}[\nu/N]$, and so $\big\langle\langle a,b\rangle,\langle c,d\rangle\big\rangle \in c_1\,[(p\,;\sigma_{K_1}) \cdot q]^{\mathfrak{A}}\,[\nu/N]$. Now $\big\langle\langle a,b\rangle,\langle c,d\rangle\big\rangle \in c_0\,[(\sigma_{K_2}^{\smile}\,;p^{\smile}) \cdot q^{\smile}]^{\mathfrak{A}}\,[\nu/N]$ can be proved similarly. The proof of the inclusion $\supseteq$ is a straightforward computation.

For (ii): Take some $x \in K|_{1^{\mathfrak{A}}}$. By (i), we have $x = \langle u,v\rangle \in \sigma_K^{\mathfrak{A}}[\nu/N] \subseteq 1^{\mathfrak{A}}$, and so $\big\langle\langle u,v\rangle,u\big\rangle \in p^{\mathfrak{A}}$, $\big\langle\langle u,v\rangle,v\big\rangle \in q^{\mathfrak{A}}$ and $\big\langle\langle u,v\rangle,\langle u,v\rangle\big\rangle \in 1'^{\mathfrak{A}}$ all follow by (qp1). Therefore, $\langle x,x\rangle \in \big(c_1[(p\,;\sigma_K)\cdot q]\cdot 1'\big)^{\mathfrak{A}}[\nu/N]$ as required. The inclusion $\supseteq$ is an easy consequence of (i). $\square$

Our next aim is to 'force' $K|_{1^{\mathfrak{A}}}$ to contain only 'pure' number tuples, that is, those that have numbers 'built up from' the same zero at each of their coordinates. To begin with, we represent every $K \in Pow(N)$ as a tree whose branches are words of p s and q s (instead of a tedious definition, see Fig. 2 for an example).

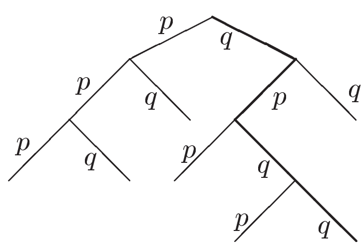

Fig. 2 $\big((N \times N) \times N\big) \times \big(\big(N \times (N \times N)\big) \times N\big)$ as a tree.

Let $br(K)$ denote the set of all branches (taken from root to leaf) of K (for instance, $br\,\Big(\big((N \times N) \times N\big) \times \big(\big(N \times (N \times N)\big) \times N\big)\Big)$ consists of ppp, ppq, pq, qpp, $qpqp$, $qpqq$, and qq). For every $r \in br(K)$ and $x \in K$, let x_r denote the element of N that is 'at the end' of branch r (e.g., if $K = \big((N \times N) \times N\big) \times \big(\big(N \times (N \times N)\big) \times N\big)$, $r = qpqp$ and $x = \big\langle\big\langle\langle a,b\rangle,c\big\rangle,\big\langle\big\langle d,\langle e,f\rangle\big\rangle,g\big\rangle\big\rangle$ then $x_r = e$). We define a $K \to K$ 'r-successor' function that increases x_r by one, but leaves $x \in K$ unchanged otherwise. For all $x \in K$, $r,s \in br(K)$, we let

$$succ^r(x)_s \stackrel{\text{def}}{=} \begin{cases} x_s & \text{if } s \neq r\,, \\ u^{(k+1)} & \text{if } s = r \text{ and } x_r = u^{(k)}\,. \end{cases}$$

We want to define this r-successor function in our algebras. To this end, for any word r of ps and qs we define a t^{ra}_{pq}-term ϱ^r by taking $\varrho^\emptyset \stackrel{\text{def}}{=} p \cdot q$ and

$$\varrho^r \stackrel{\text{def}}{=} \begin{cases} (p\,;\varrho^s)\,;p^\smile \cdot q\,;q^\smile & \text{if } r \text{ is } p \text{ followed by } s\,, \\ (q\,;\varrho^s)\,;q^\smile \cdot p\,;p^\smile & \text{if } r \text{ is } q \text{ followed by } s\,. \end{cases}$$

The following can be proved by induction on $\sqsubset$:

$$\text{For all } K \in Pow(N), r \in br(K), x \in K, \text{ if } \langle y, x\rangle \in \varrho^{r\mathfrak{A}} \text{ then } y = succ^r(x). \tag{4}$$

Indeed, if $K = N$ and $\langle y, u^{(k)}\rangle \in (p \cdot q)^{\mathfrak{A}}$ then $y = \langle u^{(k)}, u^{(k)}\rangle = u^{(k+1)}$ follows. Now suppose that $K = K_1 \times K_2$ and $r \in br(K)$ begins with p followed by some $s \in br(K_1)$. Then $\varrho^r = (p\,;\varrho^s)\,;p^\smile \cdot q\,;q^\smile$. If $x \in K$ and $\langle y, x\rangle \in (\varrho^r)^{\mathfrak{A}}$ then $y = \langle a, b\rangle, x = \langle c, b\rangle$, and $\big\langle\langle a, b\rangle, c\big\rangle \in (p\,;\varrho^s)^{\mathfrak{A}}$, for some $c \in K_1$. Thus $\langle a, c\rangle \in \varrho^{s\mathfrak{A}}$, so by the induction hypothesis, we have $a = succ^s(c)$. Therefore,

$$y = \langle a, b\rangle = \langle succ^s(c), b\rangle = succ^r(\langle c, b\rangle) = succ^r(x).$$

The case when r begins with q followed by some $s \in br(K_2)$ can be proved analogously.

We can single out the special number tuples from K as follows. For $u \in Z_N$, we call a tuple $x \in K$ *u-pure* if for every $r \in br(K)$ there is some $k \in \omega$ such that $x_r = u^{(k)}$. Let K_u denote the set of all u-pure elements in K, and let

$$K^\star \stackrel{\text{def}}{=} \bigcup_{u \in Z_N} K_u,$$

that is, $K^\star$ is the set of all *pure* elements in K.

Now recall the term id_K from (3) and Lemma 3.1. Let $\Psi_{\vec{K}}(\nu)$ be the following conjunction of t^{ra}_{pq}- type equations:

$$\bigwedge_{r \in br(K)} 1\,;\mathsf{id}_K \le 1\,;\varrho^r\,.$$

The next lemma shows how we can force pure number-tuples to be in the unit.

Lemma 3.2 *Let $K \in Pow(N)$. If*

$$\mathfrak{A} \models (\text{qp1}) \wedge \big(\Phi^1_\omega \wedge \bigwedge_{L \sqsubset K} \Psi_{\vec{L}}\big)[\nu/N]$$

then $K^\star \subseteq K|_{1^\mathfrak{A}}$ holds, that is, the pure elements of K 'hereditarily' belong to $1^\mathfrak{A}$.

P r o o f. We prove the lemma by induction on $\sqsubset$. To begin with, we have $N^\star = N = N|_{1^\mathfrak{A}}$ by Lemma 2.2(vi).

Now let $K = K_1 \times K_2$ and suppose that $L^\star \subseteq L|_{1^\mathfrak{A}}$ holds, for every $L \sqsubset K$. For all $L \in Pow(N)$ and $u \in Z_N$, let $\bar{0}^u_L$ denote the 'all-$u^{(0)}$' element in $L_u \subseteq L^\star$ (e.g., if $L = N \times (N \times N)$ then $\bar{0}^u_L$ is $\big\langle u^{(0)}, \langle u^{(0)}, u^{(0)}\rangle\big\rangle$). First, we claim that $\bar{0}^u_K$ belongs to $K|_{1^\mathfrak{A}}$, for every $u \in Z_N$. Indeed, as $\bar{0}^u_K = \langle \bar{0}^u_{K_1}, \bar{0}^u_{K_2}\rangle$ and $\bar{0}^u_{K_i} \in K_i|_{1^\mathfrak{A}}$ for $i = 1, 2$ by the induction hypothesis, we only need to show that $\bar{0}^u_K$ belongs to $1^\mathfrak{A}$. If $K_1 = K_2 = N$ then $\bar{0}^u_K = \langle u^{(0)}, u^{(0)}\rangle = u^{(1)}$ belongs to N, and so to $1^\mathfrak{A}$. So suppose that, say, $K_1 \neq N$. Then we can again use the previous 'trick:' as $\langle u^{(0)}, u^{(0)}\rangle$ belongs not only to $N \times N$ but also to N, $\bar{0}^u_{K_1}$ belongs to $L^\star$ for some $L \sqsubset K_1$, and so $\bar{0}^u_K = \langle \bar{0}^u_{K_1}, \bar{0}^u_{K_2}\rangle$ belongs to $(L \times K_2)^\star$. As $L \times K_2 \sqsubset K$, $\bar{0}^u_K \in 1^\mathfrak{A}$ follows by the induction hypothesis.

Next, we claim that if $x \in K|_{1^\mathfrak{A}}$ for some $x \in K^\star$, then $succ^r(x) \in K|_{1^\mathfrak{A}}$ for every $r \in br(K)$. Indeed, let $x = \langle a, b\rangle$ and suppose first that r begins with p followed by some $s \in br(K_1)$. As $a \in K_1|_{1^\mathfrak{A}}$, by Lemma 3.1(ii) we have $\langle a, a\rangle \in \mathsf{id}^{\mathfrak{A}}_{K_1}[\nu/N]$. As $1^\mathfrak{A}$ is symmetric, we also have $\langle b, a\rangle \in 1^\mathfrak{A}$, and so $\langle b, a\rangle \in (1\,;\mathsf{id}_{K_1})^{\mathfrak{A}}[\nu/N]$. Now by $\Psi_{\vec{K_1}}$ we obtain that $\langle b, a\rangle \in (1\,;\varrho^s)^{\mathfrak{A}}[\nu/N]$, and so $\langle b, succ^s(a)\rangle \in 1^\mathfrak{A}$ follows by (4) above. Therefore, we have $succ^r(x) = \langle succ^s(a), b\rangle \in 1^\mathfrak{A}$ as required. The case when r begins with q followed by some $s \in br(K_2)$ is similar. □

Next, we want to exclude non-pure number-tuples from $K|_{1^{\mathfrak{A}}}$. First, we can exclude non-pure zero-pairs by the following equation:

$$\text{(e6)} \quad \big((p\cdot q)\ ;\ \zeta\ ;(p\cdot q)^{\smile}\big)\,;1\,;\big((p\cdot q)\ ;\ \zeta\ ;(p\cdot q)^{\smile}\big)\leq 1'$$

Recall from Section 2 that $\zeta^{\mathfrak{A}}[\nu/N]=\{\langle u,u\rangle : u\in Z_N\}$. Now it is straightforward to see that if $\mathfrak{A}\models(\Phi^1_\omega\wedge\text{(e6)})[\nu/N]$ then

$$\text{for all } u,v\in Z_N,\text{ if } u\neq v \text{ then } \langle\langle u,u\rangle,\langle v,v\rangle\rangle\notin 1^{\mathfrak{A}}. \tag{5}$$

Next, we show that we can use the terms ϱ^r to simulate the 'r-predecessor' function as well. Namely, suppose that $\mathfrak{A}\models\Phi^1_\omega[\nu/N]$. The proof of the following claim is similar to that of (4):

$$\begin{gathered}\text{For all } K\in Pow(N), r\in br(K), y\in K,\\ \text{if } y_r\neq u^{(0)} \text{ for any } u\in Z_N \text{ and } \langle y,x\rangle\in\varrho^{r\mathfrak{A}} \text{ then } x\in K \text{ and } y=succ^r(x).\end{gathered}\tag{6}$$

Now, for every $K\in Pow(N)$, recall the term σ_K from (2) and Lemma 3.1. For every $r\in br(K)$, let σ^r_K be the same as σ_K but with the variable ν 'at its rth place' being replaced by $\nu\cdot\mathsf{Do}\,p\cdot\mathsf{Do}\,q$ (that is, by the domain of the predecessor function). For instance, if $K=(N\times N)\times N$ and r is pq then σ^r_K is

$$c_1\Big[\Big(p\,;\big(c_1\left[(p\,;\nu)\cdot q\right]\cdot c_0\left[(\nu\cdot\mathsf{Do}\,p\cdot\mathsf{Do}\,q)^{\smile};p^{\smile})\cdot q^{\smile}\right]\big)\Big)\cdot q\Big]\ \cdot\ c_0\big[(\nu^{\smile};p^{\smile})\cdot q^{\smile}\big]\,.$$

Then let

$$\mathsf{id}^r_K\ \stackrel{\text{def}}{=}\ c_1[(p\,;\sigma^r_K)\cdot q]\cdot 1'\,,$$

and let $\Psi^{\leftarrow}_K(\nu)$ be the following conjunction of equations:

$$\bigwedge_{r\in br(K)}\mathsf{id}^r_K\,;1\leq\varrho^r\,;1\,.$$

Finally, let Φ^2_ω be the conjunction of Φ^1_ω, (qp1) and (e6). Now Lemma 2.1, together with the fact that the unit $U\times U$ of $\mathfrak{Tpa}_u$ is closed under forming ordered pairs, gives us the following:

Lemma 3.3 $\mathfrak{Tpa}_u\models(\Phi^2_\omega\wedge\Psi^{\rightarrow}_K\wedge\Psi^{\leftarrow}_K)[\nu/N_u]$, *for every* $K\in Pow(N_u)$.

On the other hand, we have:

Lemma 3.4 *For every* $K\in Pow(N)$, *if*

$$\mathfrak{A}\models\big(\Phi^2_\omega\wedge\bigwedge_{L\sqsubset K}\Psi^{\rightarrow}_L\wedge\Psi^{\leftarrow}_L\big)[\nu/N]$$

then $K|_{1^{\mathfrak{A}}}\subseteq K^\star$.

P r o o f. We prove by induction on $\sqsubset$ that, for all $K\in Pow(N)$, $x\in K-K^\star$, we have $x\notin K|_{1^{\mathfrak{A}}}$. For $K=N$, we have $N=N^\star$ by Lemma 2.2(vi), so the statement follows.

Now let $K=K_1\times K_2$ and suppose that if $x\in L-L^\star$ then $x\notin L|_{1^{\mathfrak{A}}}$, for every $L\sqsubset K$. We call an element $x\in K$ an *all-zero* if $x_r=u^{(0)}$ for all $r\in br(K)$ (it can be a different u for different r s). Recall that $\bar{0}^u_K$ denotes the all-zero in K where the u *is* the same everywhere. First, we claim that if x is an all-zero in $K-K^\star$ then x does not belong to $1^{\mathfrak{A}}$. Indeed, let $x=\langle a,b\rangle$ for some all-zeros $a\in K_1$, $b\in K_2$. There are three cases:

(1) either $a\in K_1-K_1^\star$,

(2) or $b\in K_2-K_2^\star$,

(3) or $a=\bar{0}^u_{K_1}$, $b=\bar{0}^v_{K_2}$, and $u\neq v$.

In case (1) $a \notin K_1|_{1^{\mathfrak{A}}}$ follows by the induction hypothesis, and so we have $x = \langle a, b\rangle \notin K|_{1^{\mathfrak{A}}}$. Case (2) is similar. Case (3): If $K_1 = K_2 = N$ then the statement follows from (5) above. So suppose that, say, $K_1 \neq N$. As $\langle u^{(0)}, u^{(0)}\rangle$ belongs not only to $N \times N$ but also to N, $\bar{0}^u_{K_1}$ belongs to $L^\star$ for some $L \sqsubset K_1$. Therefore,

$$\bar{0}^u_{K_1} \text{ belongs to } L|_{1^{\mathfrak{A}}}, \text{ and } \bar{0}^v_{K_2} \text{ belongs to } K_2|_{1^{\mathfrak{A}}}, \tag{7}$$

by Lemma 3.2. On the other hand, $\langle a, b\rangle = \langle \bar{0}^u_{K_1}, \bar{0}^v_{K_2}\rangle$ belongs to $L \times K_2$ but, as $u \neq v$, it does not belong to $(L \times K_2)^\star$. So $\langle a, b\rangle \notin (L \times K_2)|_{1^{\mathfrak{A}}}$, by the induction hypothesis. Therefore, by (7) $\langle a, b\rangle \notin 1^{\mathfrak{A}}$, and so it does not belong to $K|_{1^{\mathfrak{A}}}$ either.

Next, we claim that if $succ^r(x) \in K|_{1^{\mathfrak{A}}}$ for some $r \in br(K)$, then $x \in K|_{1^{\mathfrak{A}}}$. Indeed, suppose first that r begins with p followed by some $s \in br(K_1)$, that is, $succ^r(x) = \langle succ^s(a), b\rangle$ for some $succ^s(a) \in K_1|_{1^{\mathfrak{A}}}$. By Lemma 3.1(ii) we have $\langle succ^s(a), succ^s(a)\rangle \in \mathsf{id}^{s\mathfrak{A}}_{K_1}[\nu/N]$, and so $\langle succ^s(a), b\rangle \in (\mathsf{id}^s_{K_1}\,; 1)^{\mathfrak{A}}[\nu/N]$. Now by $\Psi^{\leftarrow}_{K_1}$ we obtain that $\langle succ^s(a), b\rangle \in (\varrho^s\,;1)^{\mathfrak{A}}[\nu/N]$, and so $x = \langle a, b\rangle \in 1^{\mathfrak{A}}$ follows by (6), as required. The case when r begins with q followed by some $s \in br(K_2)$ is similar. □

As a consequence of Lemmas 3.1, 3.2 and 3.4, we obtain that the terms σ_K define the pure tuples in the powers of N:

Lemma 3.5 *For every $K \in Pow(N)$, if*

$$\mathfrak{A} \models \big(\Phi^2_\omega \wedge \bigwedge_{L \sqsubset K} \Psi^{\rightarrow}_L \wedge \Psi^{\leftarrow}_L\big)[\nu/N]$$

then

(i) $K^\star \;=\; K|_{1^{\mathfrak{A}}} \;=\; \sigma^{\mathfrak{A}}_K[\nu/N]$, *and*

(ii) $\{\langle x, x\rangle : x \in K^\star\} \;=\; \{\langle x, x\rangle : x \in K|_{1^{\mathfrak{A}}}\} \;=\; \mathsf{id}^{\mathfrak{A}}_K[\nu/N]$.

Note that we did not necessarily exclude non-pure number tuples like $\big\langle\langle v^{(5)}, w^{(3)}\rangle, u^{(2)}\big\rangle$ from the unit. It can happen that, say, $\big\langle\langle v^{(5)}, w^{(3)}\rangle, u^{(2)}\big\rangle \in 1^{\mathfrak{A}}$ but $\langle v^{(5)}, w^{(3)}\rangle \notin 1^{\mathfrak{A}}$. That is why in what follows when we deal with a number-tuple from K, then we always explicitly 'bind' it by the corresponding term σ_K.

4 Defining addition and multiplication

Now we have all the tools handy to define the addition and multiplication operations on numbers in PWAs. As they are binary functions, they can be considered as ternary relations. We intend to simulate each of them as a set of pure number triples, that is, as a subset of $\big((N \times N) \times N\big)^\star$.

Let α and μ be fresh variables and let $\Psi_{\underline{\omega}}(\nu, \alpha, \mu)$ be the conjunction of Φ^2_ω, $\Psi^{\rightarrow}_{(N\times N)\times(N\times N)}$, $\Psi^{\leftarrow}_{(N\times N)\times(N\times N)}$ and the following equations:

(e7) $\alpha \leq \sigma_{(N\times N)\times N}$

(e8) $\mathsf{Do}\,\alpha = \mathsf{id}_{N\times N}$

(e9) $\alpha^{\smile}; \alpha \leq 1'$

(e10) $p\,; \mathsf{id}_N \cdot \big[q\,; \big((p\cdot q)\,;\,\zeta\,;(p\cdot q)^{\smile}\big)\big]\,; 1 \;\leq\; \alpha$

(e11) $\big(\big[p\,; \mathsf{id}_N\,; p^{\smile} \cdot q\,; \big((p\cdot q)\,; \mathsf{id}_N\,; q^{\smile}\big)\big]\,; \alpha\big)\,;(p\cdot q)^{\smile} \;\leq\; \alpha$

(e12) $\mu \leq \sigma_{(N\times N)\times N}$

(e13) $\mathsf{Do}\,\mu = \mathsf{id}_{N\times N}$

(e14) $\mu^{\smile}; \mu \leq 1'$

(e15) $p\,; \mathsf{id}_N\,; 1 \cdot q\,; \big((p\cdot q)\,;\,\zeta\,;(p\cdot q)^{\smile}\big) \;\leq\; \mu$

(e16) $\big[\big(\big[p\,; \mathsf{id}_N\,; p^{\smile} \cdot q\,; \big((p\cdot q)\,; \mathsf{id}_N\,; q^{\smile}\big)\big]\,; \mu\big)\,; p^{\smile} \cdot p\,; q^{\smile}\big]\,; \alpha \;\leq\; \mu.$

Lemma 3.3, together with some tedious but straightforward computations, results in the following:

Lemma 4.1 *Let*

$$A_u \stackrel{\text{def}}{=} \big\{\big\langle\langle u^{(k)}, u^{(\ell)}\rangle, u^{(k+\ell)}\big\rangle : k, \ell \in \omega\big\}\,,$$
$$M_u \stackrel{\text{def}}{=} \big\{\big\langle\langle u^{(k)}, u^{(\ell)}\rangle, u^{(k*\ell)}\big\rangle : k, \ell \in \omega\big\}\,.$$

Then $\mathfrak{Tpa}_u \models \Psi_{\underline{\omega}}[\nu/N_u, \alpha/A_u, \mu/M_u]$.

On the other hand, we show that these equations force structures isomorphic to the standard model of arithmetic in any PWA. To this end, we fix some PWA $\mathfrak{A}$ (with base U) and elements $N, \mathit{Add}, \mathit{Mult}$ in $\mathfrak{A}$. Recall the notation Z_N, N_u and $succ_u$ from Section 2.

Lemma 4.2 *Suppose* $\mathfrak{A} \models \Psi_{\underline{\omega}}[\nu/N, \alpha/\mathit{Add}, \mu/\mathit{Mult}]$. *Then the following hold, for every* $u \in Z_N$*:*

(i) $\mathit{Add} \subseteq \big((N \times N) \times N\big)^\star$ *and* $\mathit{Mult} \subseteq \big((N \times N) \times N\big)^\star$,

(ii) $+_u \stackrel{\text{def}}{=} \mathit{Add} \cap \big((N_u \times N_u) \times N_u\big)$ *and* $*_u \stackrel{\text{def}}{=} \mathit{Mult} \cap \big((N_u \times N_u) \times N_u\big)$ *are total functions on* $N_u \times N_u$, *and*

(iii) $\mathcal{N}_u \stackrel{\text{def}}{=} \big\langle N_u, \langle u, u\rangle, succ_u, +_u, *_u\big\rangle$ *is isomorphic to* $\langle \omega, 0, succ, +, *\rangle$.

P r o o f. Item (i) clearly follows from (e7), (e12) and Lemma 3.5.

For (ii): As $N_u \times N_u \subseteq 1^{\mathfrak{A}}$ by Lemma 3.5, $+_u$ and $*_u$ are functions on $N_u \times N_u$ by (e9) and (e14). That each of them has $N_u \times N_u$ as its domain follows from (e8) and (e13).

For (iii): First, we claim that for all $u \in Z_N, n, m \in \omega$,

$$u^{(n)} +_u u^{(m)} = u^{(n+m)}. \tag{8}$$

Indeed, this is proved by induction on m. For $m = 0$, as Lemma 3.5 implies that $(N \times N)^\star$ and $\big((N \times N) \times N\big)^\star$ are included in $1^{\mathfrak{A}}$, we have

$$\big\langle\langle u^{(n)}, u^{(0)}\rangle, u^{(n)}\big\rangle \in \big(p\,; \mathsf{id}_N \cdot \big[q\,; \big((p \cdot q)\ ;\ \zeta\ ; (p \cdot q)^\smile\big)\big]\,; 1\big)^{\mathfrak{A}}[\nu/N],$$

and so $\big\langle\langle u^{(n)}, u^{(0)}\rangle, u^{(n)}\big\rangle \in \mathit{Add}$ by (e10). Now assume that (8) holds for m. Since the set $\big((N{\times}N){\times}(N{\times}N)\big)^\star$ is also included in $1^{\mathfrak{A}}$, we have

$$\big\langle\langle u^{(n)}, u^{(m+1)}\rangle, \langle u^{(n)}, u^{(m)}\rangle\big\rangle \in \big(p\,; \mathsf{id}_N\,; p^\smile \cdot q\,; \big((p \cdot q)\,; \mathsf{id}_N\,; q^\smile\big)\big)^{\mathfrak{A}}[\nu/N],$$

and so

$$\big\langle\langle u^{(n)}, u^{(m+1)}\rangle, u^{(n+m+1)}\big\rangle \in \big(\big(\big[p\,; \mathsf{id}_N\,; p^\smile \cdot q\,; \big((p \cdot q)\,; \mathsf{id}_N\,; q^\smile\big)\big]\,; \alpha\big)\,; (p \cdot q)^\smile\big)^{\mathfrak{A}}[\nu/N, \alpha/\mathit{Add}],$$

by the induction hypothesis. Now $\big\langle\langle u^{(n)}, u^{(m+1)}\rangle, u^{(n+m+1)}\big\rangle \in \mathit{Add}$ follows by (e11).

Next, using (8) and that the appropriate u-pure tuples belong to $1^{\mathfrak{A}}$, one can prove by another induction on m that $u^{(n)} *_u u^{(m)} = u^{(n*m)}$, for all $u \in Z_N, n, m \in \omega$. Equations (e15) and (e16) are used in the proofs of $u^{(n)} *_u u^{(0)} = u^{(0)}$ and $u^{(n)} *_u u^{(m+1)} = u^{(n)} *_u u^{(m)} +_u u^{(n)}$, respectively. □

5 Encoding Diophantine equations in PWAs

We fix a class K of PWAs such that the true pairing algebra $\mathfrak{Tpa}_u$ belongs to K for some non-pair set u. In order to prove Theorem 1.1, we need to define a recursive translation of Diophantine equations to $t^{\,ra}_{\,pq}$-type quasi-equations such that for any Diophantine equation φ, φ is unsolvable in $\underline{\omega}$ iff its translation q_φ is valid in K. As we shall see below, q_φ will be of the form $\Psi_{\underline{\omega}} \to (\tau_\varphi = 0)$, for some t^{ra}_{pq}-type term τ_φ having the same variables ν, α and μ as $\Psi_{\underline{\omega}}$.

The term τ_φ is obtained from φ via the following three steps:

(1) First, we translate φ to a kind of equivalent formula of a relational similarity type where all non-logical symbols are binary predicates. Here we use the fact that, with the help of the projections, the binary functions of addition and multiplication can be expressed as not only ternary but *binary* relations. When we evaluate a Diophantine equation in $\underline{\omega}$, all variables in it range over numbers. When we turn addition and multiplication to binary relations, some variables will range over pairs of numbers.

(2) Then we use the translation of Tarski and Givant [22] to turn the resulting 'all-binary' formula to an equivalent formula having only three variables (free and bound). This again can be done because of the presence of the projections. This translation is meant to work for true pairing algebras and in general we have only PWAs with a 'relativised' unit. However, we shall see that one direction of the translation still works in arbitrary PWAs as well.

(3) Finally, we use another technique of Tarski and Givant [22] to turn the resulting 'three-variable, all-binary' formula to a t^{ra}_{pq}-type term in a 'meaning preserving' way. Again, this translation is meant to work for true pairing algebras, but as the all-binary formulas obtained in step (2) are of a special 'existential conjunctive' form, one direction continues to work in arbitrary PWAs.

5.1 Translating Diophantine equations to 'all-binary' formulas

Let t^{bin} denote the relational similarity type having the following binary predicate symbols:

$$T, P, Q, I_N, I_Z, A, \text{and } M. \tag{9}$$

We call a t^{bin}-type atomic formula $R(x, y)$ *irreflexive* if x and y are distinct variables.

We define a recursive translation which turns each Diophantine equation φ to a conjunction φ^+ of irreflexive atomic t^{bin}-type formulas. At each step of the translation below, y_1, y_2, y_3 always denote fresh, distinct variables.

- $(z = z')^+ \stackrel{\text{def}}{=} I_N(z, z')$, whenever z and z' are distinct,
- $(z = z)^+ \stackrel{\text{def}}{=} I_N(y_1, z)$,
- $(0 = z)^+ \stackrel{\text{def}}{=} I_Z(y_1, z)$,
- $(succ(t) = z)^+ \stackrel{\text{def}}{=} (t = y_1)^+ \wedge P(z, y_1) \wedge Q(z, y_1)$,
- $(t_1 + t_2 = z)^+ \stackrel{\text{def}}{=} (t_1 = y_1)^+ \wedge (t_2 = y_2)^+ \wedge A(y_3, z) \wedge P(y_3, y_1) \wedge Q(y_3, y_2)$,
- $(t_1 * t_2 = z)^+ \stackrel{\text{def}}{=} (t_1 = y_1)^+ \wedge (t_2 = y_2)^+ \wedge M(y_3, z) \wedge P(y_3, y_1) \wedge Q(y_3, y_2)$,
- $(t_1 = t_2)^+ \stackrel{\text{def}}{=} (t_1 = y_1)^+ \wedge (t_2 = y_2)^+ \wedge y_1 = y_2$.

Observe that T does not occur in φ^+ (we will need it later on), and if a variable occurs in φ then it occurs in φ^+ as well.

Now assume that $\mathfrak{A}$ is a PWA with base U and $\mathfrak{A} \models \Psi_{\underline{\omega}}[\nu/N, \alpha/Add, \mu/Mult]$ for some $N, Add, Mult$ in $\mathfrak{A}$. We define a t^{bin}-type structure by taking

$$\mathfrak{A}^{N,Add,Mult} \stackrel{\text{def}}{=} \langle U, 1^{\mathfrak{A}}, P^{\mathfrak{A}}, Q^{\mathfrak{A}}, Id_N, \{\langle u^{(0)}, u^{(0)}\rangle : u \in Z_N\}, Add, Mult\rangle \tag{10}$$

(here the interpretations of the predicate symbols are listed following their order in (9)). $\mathfrak{A}^{N,Add,Mult}$ is meant to be a kind of 'relational counterpart' of the simulated arithmetic in $\mathfrak{A}$. However, calculations in $\mathfrak{A}^{N,Add,Mult}$ and $\mathfrak{A}$ are not completely analogous. Quantifiers in $\mathfrak{A}^{N,Add,Mult}$ range over U, so all pairs in $U \times U$ are 'available.' On the other hand, only pairs in $1^{\mathfrak{A}}$ are 'available' in $\mathfrak{A}$ and $1^{\mathfrak{A}}$ is typically a proper subset of $U \times U$. Yet, the following lemma shows that the translation $^+$ works.

Lemma 5.1 *Let $\mathfrak{A}$ be a* PWA *such that $\mathfrak{A} \models \Psi_{\underline{\omega}}[\nu/N, \alpha/Add, \mu/Mult]$ for some $N, Add, Mult$ in $\mathfrak{A}$. Let φ be a Diophantine equation with variables $z_1, \ldots, z_k$, and let $y_1, \ldots, y_\ell$ be those variables in φ^+ that do not occur in φ. Then the following hold:*

(i) *If* $\mathfrak{A}^{N,Add,Mult} \models \varphi^+[z_1/a_1,\dots,z_k/a_k,y_1/b_1,\dots,y_\ell/b_\ell]$ *then there exists some* $u \in Z_N$ *such that, for all* $i = 1,\dots,k$, $j = 1,\dots,\ell$,

- $a_i \in N_u$, *and*
- $b_j \in N_u \times N_u$, *whenever* $A(y_j,x)$ *or* $M(y_j,x)$ *occurs in* φ^+ *for some variable* x, *and* $b_j \in N_u$ *otherwise.*

(ii) *For all* $n_1,\dots,n_k \in \omega$, $u \in Z_N$,

$$\underline{\omega} \models \varphi[z_1/n_1,\dots,z_k/n_k] \quad \textit{iff} \quad \mathfrak{A}^{N,Add,Mult} \models \exists y_1 \dots \exists y_\ell \varphi^+[z_1/u^{(n_1)},\dots,z_k/u^{(n_k)}]\,.$$

(iii) $\underline{\omega} \models \exists z_1 \dots \exists z_k \varphi$ *iff* $\mathfrak{A}^{N,Add,Mult} \models \exists z_1 \dots \exists z_k \exists y_1 \dots \exists y_\ell \varphi^+$.

P r o o f. Item (i) is proved by induction along the definition of φ^+, using Lemma 4.2. Another induction plus (i) and Lemma 4.2 prove (ii). Item (iii) follows from (i) and (ii). □

5.2 Using three variables only

The Tarski–Givant [22] translation turns any t^{bin}-type sentence ψ to a t^{bin}-type sentence that is equivalent to ψ in true pairing algebras and has only three variables v_0, v_1, v_2. Below we go through the steps of this translation and see that, when applied to formulas like φ^+ above, one of its directions works in arbitrary PWAs as well.

To begin with, by renaming bound variables we may assume that the sentence $\exists\bar{z}\exists\bar{y}\varphi^+$ we obtained in the previous subsection is of the form

$$\exists v_0 \exists v_1 \exists \bar{x} \varphi^+, \tag{11}$$

where the variables $x_1,\dots,x_n$ in $\bar{x}$ are all distinct from each other and from v_0, v_1 and v_2. The idea is to get rid of the variables in $\bar{x}$ by using v_1 to represent $\langle \dots \langle\langle v_1, x_1\rangle, x_2\rangle \dots, x_n\rangle$. The translation makes x_i disappear one by one, using the projections and v_0, v_2 as auxiliary variables. To this end, fix some $i = 1,\dots,n$ and suppose that $R(z,w)$ is an irreflexive atomic t^{bin}-type formula having variables from $v_0, v_1, v_2, x_i, \dots, x_n$ Then the t^{bin}-type formula

$$R(z,w)^{v_1 \leftarrow \langle v_1, x_i\rangle}$$

(having variables from $v_0, v_1, v_2, x_{i+1}, \dots, x_n$) is recursively defined as follows:

- $R(z,w)^{v_1 \leftarrow \langle v_1, x_i\rangle} \overset{\text{def}}{=} R(z,w)$, if $z, w \notin \{v_1, x_i\}$
- $R(v_1,w)^{v_1 \leftarrow \langle v_1, x_i\rangle} \overset{\text{def}}{=} \exists v_2\big(R(v_2,w) \land P(v_1,v_2)\big)$, if $w \notin \{v_2, x_i\}$
- $R(z,v_1)^{v_1 \leftarrow \langle v_1, x_i\rangle} \overset{\text{def}}{=} \exists v_2\big(R(z,v_2) \land P(v_1,v_2)\big)$, if $z \notin \{v_2, x_i\}$
- $R(x_i,w)^{v_1 \leftarrow \langle v_1, x_i\rangle} \overset{\text{def}}{=} \exists v_2\big(R(v_2,w) \land Q(v_1,v_2)\big)$, if $w \notin \{v_1, v_2\}$
- $R(z,x_i)^{v_1 \leftarrow \langle v_1, x_i\rangle} \overset{\text{def}}{=} \exists v_2\big(R(z,v_2) \land Q(v_1,v_2)\big)$, if $z \notin \{v_1, v_2\}$
- $R(v_1,v_2)^{v_1 \leftarrow \langle v_1, x_i\rangle} \overset{\text{def}}{=} \exists v_0\big(R(v_0,v_2) \land P(v_1,v_0)\big)$
- $R(v_2,v_1)^{v_1 \leftarrow \langle v_1, x_i\rangle} \overset{\text{def}}{=} \exists v_0\big(R(v_2,v_0) \land P(v_1,v_0)\big)$
- $R(x_i,v_2)^{v_1 \leftarrow \langle v_1, x_i\rangle} \overset{\text{def}}{=} \exists v_0\big(R(v_0,v_2) \land Q(v_1,v_0)\big)$
- $R(v_2,x_i)^{v_1 \leftarrow \langle v_1, x_i\rangle} \overset{\text{def}}{=} \exists v_0\big(R(v_2,v_0) \land Q(v_1,v_0)\big)$
- $R(v_1,x_i)^{v_1 \leftarrow \langle v_1, x_i\rangle} \overset{\text{def}}{=} \exists v_2\big[T(v_0,v_1) \land \exists v_0\big(R(v_2,v_0) \land P(v_1,v_2) \land Q(v_1,v_0)\big)\big]$

- $R(x_i, v_1)^{v_1 \leftarrow \langle v_1, x_i \rangle} \stackrel{\text{def}}{=} \exists v_2 \big[T(v_0, v_1) \wedge \exists v_0 \big(R(v_0, v_2) \wedge P(v_1, v_2) \wedge Q(v_1, v_0) \big) \big]$

(Note that in the last two cases we added the seemingly superfluous subformula $T(v_0, v_1)$ in order to achieve that every subformula of $R(z, w)^{v_1 \leftarrow \langle v_1, x_i \rangle}$ of the form $\exists v_j \psi$ has two free variables. This property will be used in the third step of our translation, see Section 5.3.)

Now we get rid of the x_i in φ^+ one by one by defining formulas φ_i, for $i \leq n$, as follows. Let $\varphi_0 \stackrel{\text{def}}{=} \varphi^+$ (with variables as in (11)) and for $i > 0$, let φ_i be obtained by simultaneously replacing each subformula in φ_{i-1} of the form $R(z, w)$ with $R(z, w)^{v_1 \leftarrow \langle v_1, x_i \rangle}$.

An inspection of this definition shows the following properties of the φ_i, for every $i \leq n$:

- φ_i has free variables $v_0, v_1, x_{i+1}, \ldots, x_n$.
- φ_i is obtained from irreflexive atomic formulas using conjunction and existential quantification.
- All quantifiers in φ_i are of the form $\exists v_j$, for $j = 0, 2$.
- Every subformula of φ_i of the form $\exists v_j \psi$ has two free variables.

In particular, only the variables v_0, v_1 and v_2 occur in φ_n, and it has v_0 and v_1 as its free variables.

As a consequence of the results in [22] we have:

$$\mathfrak{Tpa}_u^{N_u, A_u, M_u} \models \exists v_0 \exists v_1 \exists x_1 \ldots \exists x_n \varphi^+ \leftrightarrow \exists v_0 \exists v_1 \varphi_n. \tag{12}$$

However, an induction on i, together with inspecting the corresponding cases of the definition of $R(z, w)^{v_1 \leftarrow \langle v_1, x_i \rangle}$, shows that one direction holds in a more general setting:

Lemma 5.2 *Assume that $\mathfrak{A}$ is a* PWA *with base U and $\mathfrak{A} \models \Psi_{\underline{\omega}}[\nu/N, \alpha/Add, \mu/Mult]$ for some $N, Add, Mult$ in $\mathfrak{A}$. For all $0 < i \leq n$, $a_0, a_1, b_{i+1}, \ldots, b_n \in U$, if*

$$\mathfrak{A}^{N,Add,Mult} \models \varphi_i[v_0/a_0, v_1/a_1, x_{i+1}/b_{i+1}, \ldots, x_n/b_n]$$

then there exist $a, b \in U$ such that $a_1 = \langle a, b \rangle$ and

$$\mathfrak{A}^{N,Add,Mult} \models \varphi_{i-1}[v_0/a_0, v_1/a, x_i/b, x_{i+1}/b_{i+1}, \ldots, x_n/b_n].$$

As a consequence we obtain that

$$\mathfrak{A}^{N,Add,Mult} \models \exists v_0 \exists v_1 \varphi_n \to \exists v_0 \exists v_1 \exists x_1 \ldots \exists x_n \varphi^+.$$

5.3 Translation to relation algebra terms

The second part of the Tarski–Givant [22] translation turns any three-variable t^{bin}-type formula having v_0 and v_1 as its free variables to a t^{ra}_{pq}-type term. As our formula φ_n above is of a special kind, below we discuss a special case of this translation (see also Németi [15] on the general translation, whose presentation is more similar to ours).

We call a t^{bin}-type formula ψ *special* if the following hold:

- ψ contains (as free and bound) only the variables v_0, v_1, v_2.
- ψ is obtained from irreflexive atomic formulas using conjunction and existential quantification.
- Every subformula of ψ of the form $\exists v_j \chi$ has two free variables.

Clearly, any subformula of a special formula is special as well. By definition, any special formula is a (possibly one-element) conjunction of its *conjuncts*, that is, of special formulas that are not conjunctions and that have two free variables. For any special formula ψ, define ψ^{01} as the conjunction of those conjuncts of ψ whose free variables are v_0 and v_1. If there is no such conjunct then let $\psi^{01} \stackrel{\text{def}}{=} T(v_0, v_1)$. One can define ψ^{02} and ψ^{21} similarly.

Now suppose that $\mathfrak{A}$ is a PWA with base U and $\mathfrak{A} \models \Psi_{\underline{\omega}}[\nu/N, \alpha/Add, \mu/Mult]$ for some $N, Add, Mult$ in $\mathfrak{A}$. Recall the t^{bin}-type structure $\mathfrak{A}^{N,Add,Mult}$ from (10). As T is interpreted in this structure as $1^{\mathfrak{A}}$, it is easy to see that

$$\mathfrak{A}^{N,Add,Mult} \models \psi^{01} \wedge \psi^{02} \wedge \psi^{21} \to \psi, \tag{13}$$

but the other direction might not hold. On the other hand, as in $\mathfrak{Tpa}_u$ all pairs over its base are available, we clearly have

$$\mathfrak{Tpa}_u^{N_u, A_u, M_u} \models \psi \leftrightarrow (\psi^{01} \wedge \psi^{02} \wedge \psi^{21}). \tag{14}$$

Let π be a permutation of the set $\{0,1,2\}$, and let ψ be a special formula. We let the formula $\pi\psi$ be obtained by simultaneously replacing every occurrence of v_i (free and bound as well) in ψ with $v_{\pi i}$ ($i = 0,1,2$). Clearly, $\pi\psi$ is also a special formula. It is not hard to see (cf. e.g. [22, item (ii), p.72]) that, for all t^{bin}-type structures $\mathfrak{M} = (U, \dots)$ and all a_0, a_1, a_2 in U,

$$\mathfrak{M} \models \psi[v_0/a_0, v_1/a_1, v_2/a_2] \qquad \text{iff} \qquad \mathfrak{M} \models \pi\psi[v_{\pi 0}/a_{\pi 0}, v_{\pi 1}/a_{\pi 1}, v_{\pi 2}/a_{\pi 2}]\,. \tag{15}$$

Now we define a recursive translation which turns any special formula ψ having free variables v_0 and v_1 to a t_{pq}^{ra}-type term $trm(\psi)$ having variables from ν, α, μ as follows:

- $trm(v_0 = v_1) = trm(v_1 = v_0) \stackrel{\text{def}}{=} 1'$,
- $trm(T(v_0, v_1)) = trm(T(v_1, v_0)) \stackrel{\text{def}}{=} 1$,
- $trm(P(v_0, v_1)) \stackrel{\text{def}}{=} p$ and $trm(P(v_1, v_0)) \stackrel{\text{def}}{=} p^{\smile}$,
- $trm(Q(v_0, v_1)) \stackrel{\text{def}}{=} q$ and $trm(Q(v_1, v_0)) \stackrel{\text{def}}{=} q^{\smile}$,
- $trm(I_N(v_0, v_1)) = trm(I_N(v_1, v_0)) \stackrel{\text{def}}{=} \mathsf{id}_N$,
- $trm(I_Z(v_0, v_1)) = trm(I_Z(v_1, v_0)) \stackrel{\text{def}}{=} (p \cdot q)\ ;\big(\nu - (\mathsf{Do}\,p \cdot \mathsf{Do}\,q)\big)\ ; (p \cdot q)^{\smile}$,
- $trm(A(v_0, v_1)) \stackrel{\text{def}}{=} \alpha$ and $trm(A(v_1, v_0)) \stackrel{\text{def}}{=} \alpha^{\smile}$,
- $trm(M(v_0, v_1)) \stackrel{\text{def}}{=} \mu$ and $trm(M(v_1, v_0)) \stackrel{\text{def}}{=} \mu^{\smile}$,
- $trm(\psi_1 \wedge \psi_2) \stackrel{\text{def}}{=} trm(\psi_1) \cdot trm(\psi_2)$,
- $trm(\exists v_2 \chi) \stackrel{\text{def}}{=} trm(\chi^{01}) \cdot \big(trm(\pi_{12}\chi^{02})\,; trm(\pi_{02}\chi^{21})\big)$, where π_{ij} is the permutation interchanging i and j.

Again, it follows from the results of [22] that, for all special formulas ψ having free variables v_0, v_1 and for all a_0, a_1 in the base of $\mathfrak{Tpa}_u$,

$$\mathfrak{Tpa}_u^{N_u, A_u, M_u} \models \psi[v_0/a_0, v_1/a_1] \qquad \text{iff} \qquad \langle a_0, a_1\rangle \in trm(\psi)^{\mathfrak{Tpa}_u}[\nu/N_u, \alpha/A_u, \mu/M_u]\,. \tag{16}$$

The following lemma shows that one direction of (16) holds for arbitrary PWAs:

Lemma 5.3 *Assume that $\mathfrak{A}$ is a PWA with base U and $\mathfrak{A} \models \Psi_{\underline{\omega}}[\nu/N, \alpha/Add, \mu/Mult]$ for some $N, Add, Mult$ in $\mathfrak{A}$. Then, for all special formulas ψ having free variables v_0, v_1, and for all $a_0, a_1 \in U$,*

$$\textit{if} \quad \langle a_0, a_1\rangle \in trm(\psi)^{\mathfrak{A}}[\nu/N, \alpha/Add, \mu/Mult] \quad \textit{then} \quad \mathfrak{A}^{N,Add,Mult} \models \psi[v_0/a_0, v_1/a_1]\,.$$

P r o o f. The proof is by induction on the number of quantifiers in ψ. The cases of atomic formulas and conjunctions are straightforward. So assume that ψ is of form $\exists v_2\chi$, and

$$\langle a_0, a_1\rangle \in trm(\exists v_2\chi)^{\mathfrak{A}}[\nu/N, \alpha/Add, \mu/Mult] = \\ = \big(trm(\chi^{01})\cdot\big[trm(\pi_{12}\chi^{02})\,;trm(\pi_{20}\chi^{21})\big]\big)^{\mathfrak{A}}[\nu/N, \alpha/Add, \mu/Mult].$$

Then, we have

$$\begin{aligned}(\exists a_2 \in U)\quad &\langle a_0, a_1\rangle \in trm(\chi^{01})^{\mathfrak{A}}[\nu/N, \alpha/Add, \mu/Mult],\\ &\langle a_0, a_2\rangle \in trm(\pi_{12}\chi^{02})^{\mathfrak{A}}[\nu/N, \alpha/Add, \mu/Mult], \quad \text{and}\\ &\langle a_2, a_1\rangle \in trm(\pi_{20}\psi^{21})^{\mathfrak{A}}[\nu/N, \alpha/Add, \mu/Mult].\end{aligned}$$

Therefore, by the induction hypothesis,

$$\begin{aligned}(\exists a_2 \in U)\quad &\mathfrak{A}^{N,Add,Mult} \models \chi^{01}[v_0/a_0, v_1/a_1],\\ &\mathfrak{A}^{N,Add,Mult} \models \pi_{12}\chi^{02}[v_0/a_0, v_1/a_2],\\ &\mathfrak{A}^{N,Add,Mult} \models \pi_{20}\chi^{21}[v_0/a_2, v_1/a_1],\end{aligned}$$

and so by (15) and (13),

$$(\exists a_2 \in U)\quad \mathfrak{A}^{N,Add,Mult} \models \chi[v_0/a_0, v_1/a_1, v_2/a_2],$$

as required. □

P r o o f o f T h e o r e m 1.1. Finally, we can put together the required recursive translation of Diophantine equations to t_{pq}^{ra}-type quasi-equations. Assume that we are given some Diophantine equation φ, having variables $z_1, \ldots, z_k$. Take the formula φ_n, as defined in Section 5.2. Then let

$$\tau_\varphi \stackrel{\text{def}}{=} trm(\varphi_n)\,.$$

Now let K be a class of PWAs such that the true pairing algebra $\mathfrak{Tpa}_u$ belongs to K for some non-pair set u. We claim that

$$\varphi \text{ is unsolvable in } \underline{\omega} \qquad \text{iff} \qquad \mathsf{K} \models \Psi_{\underline{\omega}} \to (\tau_\varphi = 0)\,.$$

Indeed, assume first that φ is unsolvable, that is, $\underline{\omega} \not\models \exists z_1 \ldots \exists z_k\varphi$. Let $\mathfrak{A} \in \mathsf{K}$ and $N, Add, Mult$ in $\mathfrak{A}$ be such that $\mathfrak{A} \models \Psi_{\underline{\omega}}[\nu/N, \alpha/Add, \mu/Mult]$. By Lemma 5.1 we obtain that $\mathfrak{A}^{N,Add,Mult} \not\models \exists v_0\exists v_1\exists x_1 \ldots \exists x_n\varphi^+$. Therefore, Lemma 5.2 gives us $\mathfrak{A}^{N,Add,Mult} \not\models \exists v_0\exists v_1\varphi_n$. By Lemma 5.3, we obtain $\tau_\varphi^{\mathfrak{A}}[\nu/N, \alpha/Add, \mu/Mult] = \emptyset$, as required.

For the other direction, suppose that φ is solvable, that is, $\underline{\omega} \models \exists z_1 \ldots \exists z_k\varphi$. Take the true pairing algebra $\mathfrak{Tpa}_u$ and the elements N_u, A_u and M_u in it. On the one hand, we have $\mathfrak{Tpa}_u \models \Psi_{\underline{\omega}}[\nu/N_u, \alpha/A_u, \mu/M_u]$, by Lemma 4.1. On the other, by Lemma 5.1 we obtain $\mathfrak{Tpa}_u^{N_u,A_u,M_u} \models \exists v_0\exists v_1\exists x_1 \ldots \exists x_n\varphi^+$. Therefore, $\tau_\varphi^{\mathfrak{Tpa}_u}[\nu/N_u, \alpha/A_u, \mu/M_u] \neq \emptyset$ by (12), (14) and (16). As $\mathfrak{Tpa}_u \in \mathsf{K}$, this implies $\mathsf{K} \not\models \Psi_{\underline{\omega}} \to (\tau_\varphi = 0)$. □

P r o o f o f T h e o r e m 1.2. Here we use $x + y$ as a shorthand for $-(-x \cdot -y)$. It is easy to see that $d(x) \stackrel{\text{def}}{=} x + D(x)$ is a unary discriminator term in PWADs, that is, for every PWAD $\mathfrak{A}$ and every element X in $\mathfrak{A}$,

$$d^{\mathfrak{A}}(X) = X \cup D^{\mathfrak{A}}(X) = \begin{cases} 1^{\mathfrak{A}} & \text{if } X \neq \emptyset,\\ 0^{\mathfrak{A}} & \text{else.}\end{cases}$$

Now, as usual in discriminator classes, one can effectively turn any quasi-equation to an equivalent equation over PWADs using the following observations. An equation of the form $\sigma = \tau$ is equivalent to the equation $\sigma\cdot\tau + -\sigma\cdot-\tau = 1$. An inequality $\sigma \neq 1$ is equivalent to the equation $d(-\sigma) = 1$. And a conjunction of the form $(\sigma = 1) \wedge (\tau = 1)$ is equivalent to the equation $\sigma\cdot\tau = 1$. □

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