$\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$ lacks the finite model property

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Abstract

It follows from algebraic results of Maddux that every multi-modal logic L such that $[\mathbf{S5}, \mathbf{S5}, \ldots, \mathbf{S5}] \subseteq L \subseteq \mathbf{S5}^n$ is undecidable, whenever $n \geq 3$. This implies that the product logic $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$ does not have the product finite model property. Here we answer a question of Gabbay and Shehtman by showing that $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$ also lacks the 'real' finite model property (fmp). We prove that every logic L from the above interval lacks the fmp. (In algebraic setting: If V is a variety of n-dimensional diagonal-free cylindric algebras which contains all the representables then V does not have the finite algebra property.)

1 Introduction and the result

Multi-modal logics are of growing importance in many areas of computer science, artificial intelligence, knowledge representation and reasoning, and linguistics. In this paper we discuss the finite model property of *n-modal logics*: propositional multi-modal logics having finitely many unary modal operators $\Diamond_0, \ldots, \Diamond_{n-1}$ (and their duals $\Box_0, \ldots, \Box_{n-1}$). Formulas of this language, using propositional variables from some fixed countable set P, are called *n-modal formulas*. For each natural number $n > 1$, the well-known (cf. [3]) n-modal logic

$$[\underbrace{\mathbf{S5}, \mathbf{S5}, \ldots, \mathbf{S5}}_{n}]$$

is the smallest set of n-modal formulas which

(1) is closed under the rules of Substitution, Modus Ponens, and Necessitation $A/\Box_i A$, for $i < n$;

and contains, for all $i, j < n$,

(2) all propositional tautologies and formulas of the form

$$\Box_i(p \to q) \to (\Box_i p \to \Box_i q);$$

(3) the $\mathbf{S5}$-axioms for $\Box_i$: $\quad \Box_i p \to p, \quad p \to \Box_i \Diamond_i p, \quad \Box_i p \to \Box_i \Box_i p;$

(4) $\Box_i \Box_j p \leftrightarrow \Box_j \Box_i p$.

Special frames for n-modal logics are the *product frames*. Given Kripke frames $\mathfrak{F}_0 = \langle W_0, R_0 \rangle, \dots, \mathfrak{F}_{n-1} = \langle W_{n-1}, R_{n-1} \rangle$, their product is defined to be the relational structure

$$\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1} = \langle W_0 \times \cdots \times W_{n-1}, \bar{R}_0, \dots, \bar{R}_{n-1} \rangle$$

where, for each $i < n$, $\bar{R}_i$ is the following binary relation on the Cartesian product $W_0 \times \cdots \times W_{n-1}$:

$$\langle u_0, \dots, u_{n-1} \rangle \bar{R}_i \langle v_0, \dots, v_{n-1} \rangle \quad \text{iff} \quad u_i R_i v_i \;\; \text{and} \;\; u_k = v_k, \;\; \text{for } k \neq i.$$

Given Kripke complete unimodal logics L_i $(i < n)$, define the *product logic* $L_0 \times \cdots \times L_{n-1}$ as the set of all n-modal formulas which are valid in those product frames $\langle W_0, R_0 \rangle \times \cdots \times \langle W_{n-1}, R_{n-1} \rangle$ where, for each $i < n$, $\langle W_i, R_i \rangle$ is a frame for L_i. This way one can obtain new logics, e.g., for $n \geq 3$, the product logic $\mathbf{S5}^n$—being non-finitely axiomatizable by [6]—is different from $[\mathbf{S5}, \mathbf{S5}, \dots, \mathbf{S5}]$. Though, it is routine to check that $[\mathbf{S5}, \mathbf{S5}, \dots, \mathbf{S5}] \subseteq \mathbf{S5}^n$ holds. Also, it is not hard to see that $\mathbf{S5}^n$ is already determined by products of rooted (i.e., universal) **S5**-frames.

From now on, we always let $n \geq 3$. An n-modal logic L has the *finite model property* (*fmp*) if for any n-formula $\varphi \notin L$ there is a finite model $\mathfrak{M}$ such that $\mathfrak{M} \models L$ and $\mathfrak{M} \not\models \varphi$. Note that it is the same as requiring the existence of a finite frame $\mathfrak{F}$ with $\mathfrak{F} \models L$ and $\mathfrak{F} \not\models \varphi$ (see e.g. [2, Thm.8.47]). L has the *product fmp* if for any $\varphi \notin L$ there is a finite product frame such that $\mathfrak{F} \models L$ and $\mathfrak{F} \not\models \varphi$. Of course, the product fmp implies the fmp, but not the other way round, e.g., $\mathbf{K}^n$ has the fmp (see [3]), but lacks the product fmp, for $n \geq 3$ (see [5]).

It follows from the algebraic results of [7] that every n-modal logic between $[\mathbf{S5}, \mathbf{S5}, \dots, \mathbf{S5}]$ and $\mathbf{S5}^n$ is undecidable (see also section 2 below for the technique of [7]). This implies that the finitely axiomatizable logic $[\mathbf{S5}, \mathbf{S5}, \dots, \mathbf{S5}]$ does not have the fmp. Since $\mathbf{S5}^n$ is recursively enumerable (see, e.g., [4]) and, by the finite axiomatizability of **S5**, finite product frames for $\mathbf{S5}^n$ are recursively enumerable, it also follows that $\mathbf{S5}^n$ lacks the product fmp. However, even if a finite frame for $\mathbf{S5}^n$ is a p-morphic image of some product frame for $\mathbf{S5}^n$, this product frame cannot necessarily be chosen finite (see [5] for a counterexample). Thus, in case of $\mathbf{S5}^n$, the lack of fmp does not follow in an obvious way from the lack of product fmp.

Our main result is the following theorem:

Theorem 1. *Let L be a logic such that*

$$[\mathbf{S5}, \mathbf{S5}, \dots, \mathbf{S5}] \subseteq L \subseteq \mathbf{S5}^n.$$

Then L lacks the finite model property, whenever $n \geq 3$.

As a corollary we obtain a negative answer to question Q21 of Gabbay and Shehtman [3]:

Corollary 2. $\mathbf{S5}^3$ *does not have the fmp.*

However, it remains open whether product logics like, e.g., $\mathbf{K4}^3$, $\mathbf{S4}^3$, $\mathbf{K4} \times \mathbf{S4} \times \mathbf{S5}$ have the fmp. Answering these questions may give some insight for intriguing open decision problems of two-dimensional products of transitive logics, such as $\mathbf{S4} \times \mathbf{S4}$, $\mathbf{K4} \times \mathbf{S4}$, $\mathbf{K4} \times \mathbf{K4}$. These logics are known to be finitely axiomatizable (see [3]), thus if they had the fmp then they would be decidable. Note that these logics lack the product fmp.

2 Modal algebras

In this section we reformulate and prove Theorem 1 in an algebraic setting, see Theorem 3 below. Similarly to the unimodal case (see e.g. [2]), an *n-modal algebra* is a Boolean algebra with n unary normal operators, that is, a structure

$$\mathfrak{A} = \langle A, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, -^{\mathfrak{A}}, 1^{\mathfrak{A}}, 0^{\mathfrak{A}}, \Diamond_i^{\mathfrak{A}} \rangle_{i<n},$$

where $\langle A, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, -^{\mathfrak{A}}, 1^{\mathfrak{A}}, 0^{\mathfrak{A}} \rangle$ is a Boolean algebra and, for all $a, b \in A$, $i < n$,

$$\Diamond_i^{\mathfrak{A}}(a +^{\mathfrak{A}} b) = \Diamond_i^{\mathfrak{A}} a +^{\mathfrak{A}} \Diamond_i^{\mathfrak{A}} b \quad \text{and} \quad \Diamond_i^{\mathfrak{A}} 0^{\mathfrak{A}} = 0^{\mathfrak{A}}.$$

A *valuation to* $\mathfrak{A}$ is a function υ mapping n-modal formulas to elements of $\mathfrak{A}$ which is defined the usual natural way: it turns the propositional connectives to the Booleans and the modal operators to themselves. An n-modal formula φ is said to be *true* in an *algebraic model* $\langle \mathfrak{A}, \upsilon \rangle$ if $\upsilon(\varphi) = 1^{\mathfrak{A}}$ holds. We say that φ is *valid* or *holds* in algebra $\mathfrak{A}$ (in symbols: $\mathfrak{A} \models \varphi$) if φ is true in all algebraic models of form $\langle \mathfrak{A}, \upsilon \rangle$. For any normal n-modal logic L, $\mathsf{Alg}L$ is the class of all n-modal algebras validating all formulas of L.

The class $\mathsf{Alg}[\mathbf{S5}, \mathbf{S5}, \ldots, \mathbf{S5}]$ is a class well-known in algebraic logic, it is the class Df_n of *diagonal-free cylindric algebras of dimension n* (see [1], [4]). In other words, an n-dimensional diagonal-free cylindric algebra is an n-modal algebra where (3)–(4) of the previous section hold. To obtain $\mathsf{Alg}\mathbf{S5}^n$, observe that every Kripke frame $\mathfrak{F} = \langle W, R_0, \ldots, R_{n-1} \rangle$ for an n-modal logic gives rise to the n-modal algebra $\mathfrak{A}(\mathfrak{F})$ of all subsets of W where, for every $X \subseteq W$, $i < n$,

$$\Diamond_i^{\mathfrak{A}(\mathfrak{F})} X = \{ w \in W \ : \ \exists u \in X \text{ with } wR_iu \}.$$

This way every product $\mathfrak{F}$ of rooted (i.e., universal) $\mathbf{S5}$-frames leads to a diagonal-free cylindric algebra $\mathfrak{A}(\mathfrak{F})$ whose elements are all subsets of some Cartesian product $W_0 \times \cdots \times W_{n-1}$ and, for any such subset X, $i < n$,

$$\Diamond_i^{\mathfrak{A}(\mathfrak{F})} X = \{ \langle w_0, \ldots, w_{n-1} \rangle : \exists u \in W_i \text{ with } \langle w_0, \ldots, w_{i-1}, u, w_{i+1}, \ldots, w_{n-1} \rangle \in X \}.$$

It is easy to see that, in general, an n-modal formula is valid in a frame $\mathfrak{F}$ iff it is valid in the algebra $\mathfrak{A}(\mathfrak{F})$. Therefore, $\mathsf{Alg}\mathbf{S5}^n$ is the variety generated by the above kind of diagonal-free cylindric algebras. This variety is called in the algebraic logic literature the variety RDf_n of *representable diagonal-free cylindric algebras of dimension* n (see Andréka et al. [1], [4]).

A variety V of n-modal algebras has the *finite algebra property* if V is generated by its finite members. In other words, for all n-modal formulas φ, if $\mathfrak{A} \not\models \varphi$ for some $\mathfrak{A} \in V$ then there is some finite $\mathfrak{B} \in V$ with $\mathfrak{B} \not\models \varphi$. It is well-known (see e.g. [2]) that a logic L has the fmp iff $\mathsf{Alg}L$ has the finite algebra property. Now we are ready to reformulate Theorem 1 for diagonal-free cylindric algebras. Note that related results about cylindric algebras are in [8].

Theorem 3. *Let $n \geq 3$, and V be a variety with $\mathsf{RDf}_n \subseteq V \subseteq \mathsf{Df}_n$. Then V does not have the finite algebra property. Namely, there is an n-modal formula φ such that*

- *$\mathfrak{A} \models \varphi$, for every finite $\mathfrak{A} \in \mathsf{Df}_n$, and*
- *there is some $\mathfrak{B} \in \mathsf{RDf}_n$ with $\mathfrak{B} \not\models \varphi$.*

Proof. Consider the following quasi-equation (that is, a conjunction of finitely many equations implying one equation) Q^* of the language of semigroups (we use $\circ$ for multiplication):

$$[(x = e\circ x = x\circ e) \wedge (y = e\circ y = y\circ e) \wedge (e = e\circ e) \wedge (x\circ y = e)] \\ \rightarrow\ (y\circ x = e).$$

Claim **3.1.** *Q^* fails in some infinite semigroup.*

Proof. For instance, consider the monoid of $\omega \to \omega$ functions with composition. It is routine to check that the function $m \mapsto m+1$ has an inverse from one side but not from the other. □

Claim **3.2.** *Q^* holds in every finite semigroup.*

Proof. Assume that the antecedent of Q^* holds in some finite semigroup $\mathfrak{S}$. Since $\mathfrak{S}$ is finite, the subsemigroup of $\mathfrak{S}$ generated by x, y and e is a finite monoid with identity element e. Then in this finite monoid $x^k = x^\ell$ must hold, for some natural numbers $k < \ell$. Therefore, by x having a right-inverse, $x^{\ell-k} = e$ holds (with $\ell - k \geq 1$). Thus

$$y\circ x = x^{\ell-k}\circ y\circ x = x^{\ell-k-1}\circ x\circ y\circ x = x^{\ell-k-1}\circ x = x^{\ell-k} = e$$

holds. □

Next, using the technique of [7], we interpret semigroups in diagonal-free cylindric algebras. Take some $\mathfrak{A} \in \mathsf{Df}_n$, and $d_0, d_1 \in A$. For any $a, b \in A$, define

$$a\circ^{\mathfrak{A}}_{d_0 d_1} b \ =\ \Diamond_2^{\mathfrak{A}} \left[\Diamond_1^{\mathfrak{A}}(d_0 \cdot^{\mathfrak{A}} \Diamond_2^{\mathfrak{A}} a) \cdot^{\mathfrak{A}} \Diamond_0^{\mathfrak{A}}(d_1 \cdot^{\mathfrak{A}} \Diamond_2^{\mathfrak{A}} b)\right].$$

Given propositional variables $r_0, r_1 \in P$ and n-modal formulas ψ, χ, the n-modal formula $\psi \circ_{r_0 r_1} \chi$ is defined analogously, by taking

$$\psi\circ_{r_0 r_1}\chi \ =\ \Diamond_2 \left[\Diamond_1(r_0 \wedge \Diamond_2\psi) \wedge \Diamond_0(r_1 \wedge \Diamond_2\chi)\right].$$

Now fix some propositional variables $r_0, r_1 \in P$. For any term τ of the language of semigroups, define inductively an n-modal formula τ^+ as follows:

- for any variable x, let $x^+ = p_x$, where $p_x \in P$ and $p_x \neq r_0, r_1$, and
- $(\sigma \circ \varrho)^+ = \sigma^+ \circ_{r_0 r_1} \varrho^+$.

For any quasi-equation Q of the language of semigroups of form

$$[(\tau_1 = \sigma_1) \ \wedge \dots \wedge \ (\tau_m = \sigma_m)] \ \to \ (\tau_0 = \sigma_0) \ ,$$

let φ_Q be the following n-modal formula:

$$\Box \left[(\tau_1^+ \leftrightarrow \sigma_1^+) \wedge \dots \wedge (\tau_m^+ \leftrightarrow \sigma_m^+) \wedge (r_0 \leftrightarrow \Diamond_0 r_0) \wedge (r_1 \leftrightarrow \Diamond_1 r_1) \right] \to \\ \to \left[(\tau_0^+ \circ_{r_0 r_1} \Diamond_1 q) \leftrightarrow (\sigma_0^+ \circ_{r_0 r_1} \Diamond_1 q) \right] \ ,$$

where $\Box\chi$ abbreviates $\Box_0 \Box_1 \dots \Box_{n-1}\chi$, and q is a fresh propositional variable.

Lemma 4. ([7]) *Let $\mathfrak{A} \in \mathsf{Df}_n$, and $d_0, d_1 \in A$ such that $\Diamond_0^{\mathfrak{A}} d_0 = d_0$ and $\Diamond_1^{\mathfrak{A}} d_1 = d_1$. For any $a, b \in A$, define*

$$a \sim_{d_0 d_1}^{\mathfrak{A}} b \quad \Longleftrightarrow \quad (\forall w \in A) \ a \circ_{d_0 d_1}^{\mathfrak{A}} \Diamond_1^{\mathfrak{A}} w = b \circ_{d_0 d_1}^{\mathfrak{A}} \Diamond_1^{\mathfrak{A}} w \ .$$

Then the following hold:

(i) $\sim_{d_0 d_1}^{\mathfrak{A}}$ *is a congruence of $\langle A, \circ_{d_0 d_1}^{\mathfrak{A}} \rangle$, that is, for any $a, b, c \in A$, if $a \sim_{d_0 d_1}^{\mathfrak{A}} b$ then*

$$(a \circ_{d_0 d_1}^{\mathfrak{A}} c) \sim_{d_0 d_1}^{\mathfrak{A}} (b \circ_{d_0 d_1}^{\mathfrak{A}} c) \quad \textit{and} \quad (c \circ_{d_0 d_1}^{\mathfrak{A}} a) \sim_{d_0 d_1}^{\mathfrak{A}} (c \circ_{d_0 d_1}^{\mathfrak{A}} b).$$

(ii) *The quotient algebra $\langle A, \circ_{d_0 d_1}^{\mathfrak{A}} \rangle / \sim_{d_0 d_1}^{\mathfrak{A}}$ is a semigroup.*

(iii) *If $\mathfrak{A}$ is a simple algebra then, for any quasi-equation Q of the language of semigroups,*

$$\langle A, \circ_{d_0 d_1}^{\mathfrak{A}} \rangle / \sim_{d_0 d_1}^{\mathfrak{A}} \models Q \quad \textit{implies} \quad \mathfrak{A} \models \varphi_Q.$$

Now recall the quasi-equation Q^* above.

Corollary 5. *φ_{Q^*} holds in every finite $\mathfrak{A} \in \mathsf{Df}_n$.*

Proof. Assume that there is some finite $\mathfrak{A} \in \mathsf{Df}_n$ and $d_0, d_1 \in A$ such that

$$\mathfrak{A} \not\models \varphi_{Q^*}[r_0/d_0, r_1/d_1] \ .$$

It is well-known (see [4]) that every subdirectly irreducible algebra in Df_n is simple, so we may assume that $\mathfrak{A}$ is simple. Then $\Diamond_0^{\mathfrak{A}} d_0 = d_0$ and $\Diamond_1^{\mathfrak{A}} d_1 = d_1$ must hold, otherwise the antecedent of φ_{Q^*} would be false. Thus, by Lemma 4 above, Q^* fails in the finite semigroup $\langle A, \circ_{d_0 d_1}^{\mathfrak{A}} \rangle / \sim_{d_0 d_1}^{\mathfrak{A}}$, contradicting Claim 3.2. $\Box$

Lemma 6. *φ_{Q^*} fails in some (infinite) $\mathfrak{B} \in \mathsf{RDf}_n$.*

Proof. This is the similar to the argument in [7], but we sketch it for completeness. By Claim 3.1, there is some (infinite) semigroup $\mathfrak{S} = \langle S, \circ \rangle$ where φ fails. We may assume that $\mathfrak{S}$ is a monoid. (If not then choose some $e \notin S$ and take the monoid $\mathfrak{S}^+ = \langle S \cup \{e\}, \circ' \rangle$, where $x \circ' y = x \circ y$ whenever $x, y \in S$ and $x \circ' e = e \circ' x = x$, for every $x \in S \cup \{e\}$.) Now consider the diagonal-free cylindric algebra $\mathfrak{A}_S \in \mathsf{RDf}_n$ of all subsets of S^n, and take the *diagonal elements* D_{02} and D_{12} of $\mathfrak{A}_S$ defined by

$$D_{i2} = \{\langle s_0, \dots, s_{n-1} \rangle \in S^n \ : \ s_i = s_2\} \quad (i = 0, 1).$$

Then $\Diamond_0^{\mathfrak{A}_S} D_{12} = D_{12}$ and $\Diamond_1^{\mathfrak{A}_S} D_{02} = D_{02}$ hold. Define a map h by taking, for any $s \in S$,

$$h(s) = \{\langle s_0, \dots, s_{n-1} \rangle \in S^n \ : \ s_1 = s_0 \circ s\} \ .$$

It is easy to check that h is one-one and, for all $s, t \in S$,

$$h(s \circ t) = h(s) \circ^{\mathfrak{A}_S}_{D_{12} D_{02}} h(t)$$

hold. (This h is in fact the Cayley representation of the monoid $\mathfrak{S}$, taken in the first two coordinates of $\mathfrak{A}_S$.) Now, in order to show that φ_{Q^*} fails in $\mathfrak{A}_S$, it is enough to prove that $h(s) \not\sim^{\mathfrak{A}_S}_{D_{12} D_{02}} h(t)$ whenever $s \neq t$. To this end, take some $s \neq t \in S$ and let

$$w = \{\langle s_0, \dots, s_{n-1} \rangle \in S^n \ : \ s_0 = s\} \ .$$

Let e denote the identity element of $\mathfrak{S}$. Then clearly $\Diamond_1^{\mathfrak{A}_S} w = w$, and any sequence of form $\langle e, \dots \rangle$ belongs to $h(s) \circ^{\mathfrak{A}_S}_{D_{12} D_{02}} w$ but not to $h(t) \circ^{\mathfrak{A}_S}_{D_{12} D_{02}} w$, showing that they are different. □

Finally, Corollary 5 and Lemma 6 prove Theorem 3. □

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