

Combining algebraizable logics

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Abstract

The general methodology of “algebraizing” logics (cf. [2], [4]) is used here for combining different logics. The combination of logics is represented as taking the colimit of the constituent logics in the category of algebraizable logics. The cocompleteness of this category as well as its isomorphism to the corresponding category of certain first-order theories are proved.

In this paper we translate the “combining logics” problem to the problem of “combining” certain theories of usual first order logic. We prove that the category of a special class of logics, called *algebraizable logical systems* (see Def. 1.1 below), is isomorphic to the category of the corresponding first order theories. We also show that these categories are cocomplete. Some directions in which the approach chosen can perhaps be generalized are pointed out in the last section.

1 Preliminaries

As a set theoretic framework we presume any set theory which is suitable for the foundation of category theory. For basic category theoretical notions such as category, object, morphism, small diagram, cocone, coproduct, colimit, coequalizer, etc. we follow the usage of MacLane [10].

Our terminology follows the usual standards concerning classical first-order logic and basics of universal algebra. For notions not defined but used here, see e.g. Monk [11], Burris-Sankappanavar [6].

ω denotes the set of natural numbers. An *algebraic similarity type* is a function t mapping some nonempty set into ω . An element f of the domain $\text{dom}(t)$ of t with $t(f) = k$ is called a *k-ary function symbol of type t*. *t-type algebras* are structures (in the usual sense) of the algebraic similarity type t . Throughout the paper we fix an infinite set $X = \{x_0, x_1, x_2, \dots\}$ of variables. x, y will always denote one of these variables. The sets Trm_t of *t-type terms*, Fmla_t of *t-type (first order) formulas*, having variables from X , are defined as usual. A *k-ary term* is a term containing at most k -many distinct variables. $\tau(x_{i_1}, \dots, x_{i_k})$ denotes that the variables occurring in τ are among x_{i_1}, \dots, x_{i_k} . Substitutions are functions $\sigma : X \rightarrow \text{Trm}_t$ as usual, which extend to maps from terms to terms the natural way. For any substitution σ and term $\tau(x_{i_1}, \dots, x_{i_k})$, $\sigma(\tau)$ will also be denoted by $\tau(x_{i_1}/\sigma(x_{i_1}), \dots, x_{i_k}/\sigma(x_{i_k}))$. A binary term $\Delta(x, y)$ will also be written as $x\Delta y$. $\underline{\text{Trm}}_t$ denotes the *t-type world-algebra* (absolutely free algebra) generated by set X .

We will use symbol “ \models ” for both validity (in models) and (semantical) consequence relation of standard first-order logic. For any set $\Gamma \subseteq \text{Fmla}_t$,

$$\text{Mod}_t(\Gamma) \stackrel{\text{def}}{=} \{\underline{A} : \underline{A} \text{ is a } t\text{-type algebra and } (\forall \varphi \in \Gamma) \underline{A} \models \varphi\}.$$

A *t-type quasi-equation* is a *t-type* formula of form $(\tau_1 = \tau'_1 \wedge \dots \wedge \tau_k = \tau'_k \rightarrow \tau_0 = \tau'_0)$, where $\tau_0, \tau'_0, \dots, \tau_k, \tau'_k \in \text{Trm}_t$. A *t-type quasivariety* is a class \mathbf{K} of *t-type* algebras such that $\mathbf{K} = \text{Mod}_t(\Gamma)$ for some set Γ of *t-type* quasi-equations. For any class \mathbf{K} of *t-type* algebras, $\text{Qvar}(\mathbf{K})$ denotes the *generated quasivariety* i.e., the smallest quasivariety including \mathbf{K} .

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Algebraizable logical systems defined below are the same as “algebraizable deductive systems” of Blok-Pigozzi [4], or “algebraizable 1-deductive systems” of [5], or the semantical consequence relation of “consequence compact strongly nice general logics” of Andr  ka et al [2].

DEFINITION 1.1 A pair $\mathcal{L} = \langle Cn(\mathcal{L}), \approx_{\mathcal{L}} \rangle$ is called an *algebraizable logical system* iff $Cn(\mathcal{L})$ is an algebraic similarity type and $\approx_{\mathcal{L}}$ is a binary relation between sets of $Cn(\mathcal{L})$ -type terms and $Cn(\mathcal{L})$ -type terms, satisfying conditions (1–6) below. Elements of the domain of $Cn(\mathcal{L})$ are called the *logical connectives of \mathcal{L}* . The elements of set X (of variables) are called in this context *atomic formulas* (or *propositional variables*) of \mathcal{L} . Similarly, if φ is a (k -ary) term of type $Cn(\mathcal{L})$ then φ is also called a (k -ary) *formula of \mathcal{L}* , and the set $Trm_{Cn(\mathcal{L})}$ is also called as $Fm(\mathcal{L})$ when it is regarded as the *set of all formulas of \mathcal{L}* . $\approx_{\mathcal{L}}$ is called the *consequence relation of \mathcal{L}* .

- (1) $(\forall \varphi \in Fm(\mathcal{L}))(\forall \Gamma \subseteq Fm(\mathcal{L})) \quad \varphi \in \Gamma \Rightarrow \Gamma \approx_{\mathcal{L}} \varphi$.
- (2) $(\forall \varphi \in Fm(\mathcal{L}))(\forall \Gamma, \Delta \subseteq Fm(\mathcal{L})) \quad \Gamma \subseteq \Delta \text{ and } \Gamma \approx_{\mathcal{L}} \varphi \Rightarrow \Delta \approx_{\mathcal{L}} \varphi$.
- (3) $(\forall \varphi \in Fm(\mathcal{L}))(\forall \Gamma, \Delta \subseteq Fm(\mathcal{L})) \quad \Gamma \approx_{\mathcal{L}} \varphi \text{ and } (\forall \psi \in \Gamma) \Delta \approx_{\mathcal{L}} \psi \Rightarrow \Delta \approx_{\mathcal{L}} \varphi$.
- (4) $(\forall \varphi \in Fm(\mathcal{L}))(\forall \Gamma \subseteq Fm(\mathcal{L})) \quad \Gamma \approx_{\mathcal{L}} \varphi \Rightarrow (\exists \text{ finite } \Gamma' \subseteq \Gamma) \Gamma' \approx_{\mathcal{L}} \varphi$.
- (5) $(\forall \varphi \in Fm(\mathcal{L}))(\forall \Gamma \subseteq Fm(\mathcal{L}))(\forall \text{ substitution } \sigma) \quad \Gamma \approx_{\mathcal{L}} \varphi \Rightarrow \{\sigma(\psi) : \psi \in \Gamma\} \approx_{\mathcal{L}} \sigma(\varphi)$.
- (6) There are some $m, n \in \omega$, unary formulas $\varepsilon_0, \dots, \varepsilon_{m-1}$ and $\delta_0, \dots, \delta_{m-1}$, and binary formulas $\Delta_0, \dots, \Delta_{n-1}$ of \mathcal{L} such that properties (i–v) below hold for any $\varphi, \varphi_1, \dots, \varphi_k, \psi, \psi_1, \dots, \psi_k, \chi \in Fm(\mathcal{L})$, and for any $i < n$.
 - (i) $\approx_{\mathcal{L}} \varphi \Delta_i \varphi$,
 - (ii) $\{\varphi \Delta_j \psi : j < n\} \approx_{\mathcal{L}} \psi \Delta_i \varphi$,
 - (iii) $\{\varphi \Delta_j \psi, \psi \Delta_j \chi : j < n\} \approx_{\mathcal{L}} \varphi \Delta_i \chi$,
 - (iv) $(\forall k\text{-ary } c \in \text{dom}(Cn(\mathcal{L})))$,
 $\{\varphi_1 \Delta_j \psi_1, \dots, \varphi_k \Delta_j \psi_k : j < n\} \approx_{\mathcal{L}} c(\varphi_1, \dots, \varphi_k) \Delta_i c(\psi_1, \dots, \psi_k)$,
 - (v) $(\forall s < m) \quad \{\varphi\} \approx_{\mathcal{L}} \varepsilon_s(\varphi) \Delta_i \delta_s(\varphi) \text{ and } \{\varepsilon_s(\varphi) \Delta_j \delta_s(\varphi) : s < m, j < n\} \approx_{\mathcal{L}} \varphi$.

A sequence $\langle \varepsilon_0, \dots, \varepsilon_{m-1}, \delta_0, \dots, \delta_{m-1}, \Delta_0, \dots, \Delta_{n-1} \rangle$ satisfying (6)(i–v) is called an *algebraizator for \mathcal{L}* . \square

Some simple examples for algebraizable logical systems are inconsistent logics (where $\Gamma \approx_{\mathcal{L}} \varphi$ holds for any Γ, φ), and usual propositional logic (with algebraizator $\varepsilon_0(\varphi) = (\varphi \rightarrow \varphi)$, $\delta_0(\varphi) = \varphi$ and $\varphi \Delta_0 \psi = (\varphi \leftrightarrow \psi)$). Other examples (also for non-algebraizable logical systems) can be found in e.g. Blok-Pigozzi [4], Andr  ka et al [2], [3], [14].

Notation 1.2 For any $\Gamma, \Delta \subseteq Fm(\mathcal{L})$, if $\Delta \neq \emptyset$ then $\Gamma \approx_{\mathcal{L}} \Delta \stackrel{\text{def}}{\iff} (\forall \psi \in \Delta) \Gamma \approx_{\mathcal{L}} \psi$.

We shall use $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ as an abbreviation for $\langle \varepsilon_0, \dots, \varepsilon_{m-1}, \delta_0, \dots, \delta_{m-1}, \Delta_0, \dots, \Delta_{n-1} \rangle$. Similarly, e.g. $\bar{\varepsilon}(\varphi) \bar{\Delta} \bar{\delta}(\psi)$ abbreviates the set $\{\varepsilon_i(\varphi) \Delta_j \delta_i(\psi) : i < m, j < n\}$ of formulas. Or, on the first order logic side, we write e.g. $\bar{\varepsilon}(x) = \bar{\delta}(x) \rightarrow \bar{\varepsilon}(y) = \bar{\delta}(y)$ instead of the set

$$\left\{ \bigwedge_{i < m} \varepsilon_i(x) = \delta_i(x) \rightarrow \varepsilon_j(y) = \delta_j(y) : j < m \right\}$$

of quasi-equations. Related abbreviations will also be used without further explanation. \square

DEFINITION 1.3 Let \mathcal{L} be an algebraizable logical system and let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ be an algebraizator for \mathcal{L} . For any $\Gamma \cup \{\varphi, \psi\} \subseteq Fm(\mathcal{L})$, let

$$\varphi \equiv_{\Gamma} \psi \stackrel{\text{def}}{\iff} \Gamma \approx_{\mathcal{L}} \varphi \bar{\Delta} \psi.$$

Then, by condition (6)(i–iv) of Def. 1.1, \equiv_{Γ} is a congruence relation on $Trm_{Cn(\mathcal{L})}$. Let

$$\text{Alg}(\mathcal{L}) \stackrel{\text{def}}{=} \text{Qvar}(\{Trm_{Cn(\mathcal{L})} / \equiv_{\Gamma} : \Gamma \subseteq Fm(\mathcal{L})\})$$

that is, $\text{Alg}(\mathcal{L})$ is a class of algebras (first order structures) of similarity type $Cn(\mathcal{L})$. \square

The definition of $\text{Alg}(\mathcal{L})$ does not depend on the choice of the algebraizator $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ as the following proposition shows.

Proposition 1.4 (cf. Blok–Pigozzi [4], Theorem 2.15)

Let \mathcal{L} be an algebraizable logical system and let both $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ and $\langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle$ be algebraizators for \mathcal{L} . Then for any formulas φ, ψ of \mathcal{L} ,

$$\varphi \bar{\Delta} \psi \approx_{\mathcal{L}} \varphi \bar{\Delta}' \psi \quad \text{and} \quad \varphi \bar{\Delta}' \psi \approx_{\mathcal{L}} \varphi \bar{\Delta} \psi. \quad \blacksquare$$

Thus, for any algebraizable logical system \mathcal{L} there is a uniquely determined quasivariety $\text{Alg}(\mathcal{L})$. In the other direction, there are different algebraizable logical systems with the same “corresponding” quasivariety, see e.g. Blok–Pigozzi [4], Ch.5.2.4 for an example.

The following “back and forth” theorem establishes the basic connection between a logic \mathcal{L} and its algebraic (i.e., usual first order) “translation” $\text{Alg}(\mathcal{L})$.

THEOREM 1.5 (cf. Blok–Pigozzi [4] Thms.2.4, 4.7, 4.10 and Andr  ka et al [2] Thm.3.2.1)

Let \mathcal{L} be an algebraizable logical system, and let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ be an algebraizator for \mathcal{L} . Then

(i) for any formulas $\varphi_0, \varphi_1, \dots, \varphi_k$ of \mathcal{L} ,

$$\{\varphi_1, \dots, \varphi_k\} \approx_{\mathcal{L}} \varphi_0 \iff \text{Alg}(\mathcal{L}) \models \bigwedge_{1 \leq s \leq k} \bar{\varepsilon}(\varphi_s) = \bar{\delta}(\varphi_s) \rightarrow \bar{\varepsilon}(\varphi_0) = \bar{\delta}(\varphi_0);$$

(ii) for any formulas $\tau_0, \tau_1, \dots, \tau_k, \tau'_0, \tau'_1, \dots, \tau'_k$ of \mathcal{L} ,

$$\text{Alg}(\mathcal{L}) \models \tau_1 = \tau'_1 \wedge \dots \wedge \tau_k = \tau'_k \rightarrow \tau_0 = \tau'_0 \iff \{\tau_1 \bar{\Delta} \tau'_1, \dots, \tau_k \bar{\Delta} \tau'_k\} \approx_{\mathcal{L}} \tau_0 \bar{\Delta} \tau'_0. \quad \blacksquare$$

2 The category of algebraizable logical systems

DEFINITION 2.1

(i) Let $\mathcal{L}_1, \mathcal{L}_2$ be algebraizable logical systems. A function $I : \text{dom}(Cn(\mathcal{L}_1)) \rightarrow Fm(\mathcal{L}_2)$ is called a *logic-translation of \mathcal{L}_1 into \mathcal{L}_2* iff for any k -ary connective $c \in \text{dom}(Cn(\mathcal{L}_1))$, $I(c)$ is a k -ary formula of \mathcal{L}_2 . A logic-translation always induces a function $\hat{I} : Fm(\mathcal{L}_1) \rightarrow Fm(\mathcal{L}_2)$ in the following natural way:

- for any propositional variable x , $\hat{I}(x) \stackrel{\text{def}}{=} x$;
- if c is a k -ary connective and $\varphi_0, \dots, \varphi_{k-1}$ are formulas of \mathcal{L}_1 then

$$\hat{I}(c(\varphi_0, \dots, \varphi_{k-1})) \stackrel{\text{def}}{=} I(c)(x_0/\hat{I}(\varphi_0), \dots, x_{k-1}/\hat{I}(\varphi_{k-1})).$$

\hat{I} can be extended to any set Γ of formulas of \mathcal{L}_1 by taking $\hat{I}(\Gamma) \stackrel{\text{def}}{=} \{\hat{I}(\varphi) : \varphi \in \Gamma\}$.

(ii) A logic-translation I is called an $(\mathcal{L}_1, \mathcal{L}_2)$ -*interpretation* iff

(a) for any $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L}_1)$,

$$\Gamma \approx_{\mathcal{L}_1} \varphi \implies \hat{I}(\Gamma) \approx_{\mathcal{L}_2} \hat{I}(\varphi);$$

(b) if $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ is an algebraizator for \mathcal{L}_1 then $\langle \hat{I}(\bar{\varepsilon}), \hat{I}(\bar{\delta}), \hat{I}(\bar{\Delta}) \rangle$ is an algebraizator for \mathcal{L}_2 .

(iii) We define an equivalence relation on $(\mathcal{L}_1, \mathcal{L}_2)$ -interpretations as follows.

$$I \sim J \iff (\forall \varphi \in Fm(\mathcal{L}_1)) \approx_{\mathcal{L}_2} \hat{I}(\varphi) \bar{\Delta}_2 \hat{J}(\varphi).$$

(Here $\langle \bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2 \rangle$ is an arbitrary algebraizator for \mathcal{L}_2 . By Prop. 1.4 and Def. 1.1(3), the definition of \sim does not depend on the choice of the algebraizator.)

Let $[I]$ denote the \sim -equivalence class of I .

- (iv) For any algebraizable logical system \mathcal{L} , let $id_{\mathcal{L}}$ be the logic-translation of \mathcal{L} into \mathcal{L} defined by $id_{\mathcal{L}}(c) \stackrel{\text{def}}{=} c(x_0, \dots, x_{k-1})$, for each k -ary connective $c \in \text{dom}(Cn(\mathcal{L}))$. \square

Lemma 2.2

- (i) Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ be algebraizable logical systems, let I, I' be $(\mathcal{L}_1, \mathcal{L}_2)$ -interpretations and let J, J' be $(\mathcal{L}_2, \mathcal{L}_3)$ -interpretations such that $I \sim I'$ and $J \sim J'$ hold. Then $\hat{J} \circ I$ and $\hat{J}' \circ I'$ are $(\mathcal{L}_1, \mathcal{L}_3)$ -interpretations, and $\hat{J} \circ I \sim \hat{J}' \circ I'$ (where \circ is the usual composition of functions).
- (ii) For any algebraizable logical system \mathcal{L} , $id_{\mathcal{L}}$ is an $(\mathcal{L}, \mathcal{L})$ -interpretation and for any $(\mathcal{L}, \mathcal{L}')$ -interpretation I , $\hat{I} \circ id_{\mathcal{L}} \sim I \sim \hat{id}_{\mathcal{L}'} \circ I$.

Proof. Since $(\hat{J} \circ I)^\wedge = \hat{J} \circ \hat{I}$, it is easy to check that $\hat{J} \circ I$ is an $(\mathcal{L}_1, \mathcal{L}_3)$ -interpretation. To prove (i), let $\langle \bar{\varepsilon}_i, \bar{\delta}_i, \bar{\Delta}_i \rangle$ be an algebraizator for \mathcal{L}_i ($i=1,2,3$), and let φ be an arbitrary formula of \mathcal{L}_1 . Then, by $I \sim I'$,

$$\begin{aligned} \approx_{\mathcal{L}_2} \hat{I}(\varphi) \bar{\Delta}_2 \hat{I}'(\varphi) &\implies (J \text{ is an interpretation}) \\ \approx_{\mathcal{L}_3} \hat{J}(\hat{I}(\varphi) \bar{\Delta}_2 \hat{I}'(\varphi)) &\iff \\ \approx_{\mathcal{L}_3} (\hat{J} \circ \hat{I})(\varphi) \hat{J}(\bar{\Delta}_2)(\hat{J} \circ \hat{I}')(\varphi) &\implies (\text{Prop. 1.4 and Def. 1.1(3)}) \\ \approx_{\mathcal{L}_3} (\hat{J} \circ \hat{I})(\varphi) \bar{\Delta}_3(\hat{J} \circ \hat{I}')(\varphi). \end{aligned}$$

On the other hand, by $J \sim J'$,

$$\approx_{\mathcal{L}_3} \hat{J}(\hat{I}'(\varphi)) \bar{\Delta}_3 \hat{J}'(\hat{I}'(\varphi)) \iff \approx_{\mathcal{L}_3} (\hat{J} \circ \hat{I}')(\varphi) \bar{\Delta}_3(\hat{J}' \circ \hat{I}')(\varphi).$$

Thus, by Def. 1.1(3) and (6)(iii), $\approx_{\mathcal{L}_3} (\hat{J} \circ \hat{I})(\varphi) \bar{\Delta}_3(\hat{J}' \circ \hat{I}')(\varphi)$ follows.

The proof of (ii) is obvious. ■

DEFINITION 2.3 The category **ALOG** of algebraizable logical systems is defined as follows.

$$\begin{aligned} \text{Obj}_{\mathbf{ALOG}} &\stackrel{\text{def}}{=} \{\mathcal{L} : \mathcal{L} \text{ is an algebraizable logical system}\} \\ \text{Mor}_{\mathbf{ALOG}}(\mathcal{L}_1, \mathcal{L}_2) &\stackrel{\text{def}}{=} \{[I] : I \text{ is an } (\mathcal{L}_1, \mathcal{L}_2)\text{-interpretation}\}, \\ &\text{for any } \mathcal{L}_1, \mathcal{L}_2 \in \text{Obj}_{\mathbf{ALOG}} \\ ID_{\mathcal{L}} &\stackrel{\text{def}}{=} [id_{\mathcal{L}}], \quad \text{for any } \mathcal{L} \in \text{Obj}_{\mathbf{ALOG}} \\ [J][I] &\stackrel{\text{def}}{=} [\hat{J} \circ I], \quad \text{for any } \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in \text{Obj}_{\mathbf{ALOG}}, \\ &[I] \in \text{Mor}_{\mathbf{ALOG}}(\mathcal{L}_1, \mathcal{L}_2), [J] \in \text{Mor}_{\mathbf{ALOG}}(\mathcal{L}_2, \mathcal{L}_3). \end{aligned}$$

Then, by Lemma 2.2, **ALOG** is indeed a category. \square

Now we proceed with making preparations to formulate the “algebraic” counterpart of category **ALOG**.

DEFINITION 2.4

- (i) Let t be an algebraic similarity type and let \mathbf{K} be a t -type quasivariety. Let $\varepsilon_0, \dots, \varepsilon_{m-1}, \delta_0, \dots, \delta_{n-1}$ be unary and $\Delta_0, \dots, \Delta_{n-1}$ be binary t -type terms for some $m, n \in \omega$. Then $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ is called a *deductivizator* of \mathbf{K} iff

$$\mathbf{K} \models \bar{\varepsilon}(x \bar{\Delta} y) = \bar{\delta}(x \bar{\Delta} y) \leftrightarrow x = y$$

holds.

(ii) We define an equivalence relation on deductivizators of \mathbf{K} as follows:

$$\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \simeq_{\mathbf{K}} \langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle \stackrel{\text{def}}{\iff} \mathbf{K} \models \bar{\varepsilon}(x) = \bar{\delta}(x) \leftrightarrow \bar{\varepsilon}'(x) = \bar{\delta}'(x).$$

Let $[\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}}$ denote the $\simeq_{\mathbf{K}}$ -equivalence class of $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$. \square

Proposition 2.5 *Let \mathcal{L} be an algebraizable logical system and let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle, \langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle$ be two algebraizators for \mathcal{L} . Then*

- (i) $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ and $\langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle$ are both deductivizators of $\mathbf{Alg}(\mathcal{L})$;
- (ii) (cf. Blok–Pigozzi [4], Theorem 2.15)
 $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \simeq_{\mathbf{Alg}(\mathcal{L})} \langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle$.

Proof.

- (i): By Def. 1.1(6)(v), $x\bar{\Delta}y \approx_{\mathcal{L}} \bar{\varepsilon}(x\bar{\Delta}y)\bar{\Delta}\bar{\delta}(x\bar{\Delta}y)$ and $\bar{\varepsilon}(x\bar{\Delta}y)\bar{\Delta}\bar{\delta}(x\bar{\Delta}y) \approx_{\mathcal{L}} x\bar{\Delta}y$. Thus, by Theorem 1.5(ii), $\mathbf{Alg}(\mathcal{L}) \models \bar{\varepsilon}(x\bar{\Delta}y) = \bar{\delta}(x\bar{\Delta}y) \leftrightarrow x = y$.
- (ii): By Def. 1.1(2),(6)(v), $\bar{\varepsilon}(x)\bar{\Delta}\bar{\delta}(x) \approx_{\mathcal{L}} \bar{\varepsilon}'(x)\bar{\Delta}'\bar{\delta}'(x)$ and $\bar{\varepsilon}'(x)\bar{\Delta}'\bar{\delta}'(x) \approx_{\mathcal{L}} \bar{\varepsilon}(x)\bar{\Delta}\bar{\delta}(x)$. Thus, by Def. 1.1(3) and Prop. 1.4, $\bar{\varepsilon}(x)\bar{\Delta}\bar{\delta}(x) \approx_{\mathcal{L}} \bar{\varepsilon}'(x)\bar{\Delta}\bar{\delta}'(x)$ and $\bar{\varepsilon}'(x)\bar{\Delta}\bar{\delta}'(x) \approx_{\mathcal{L}} \bar{\varepsilon}(x)\bar{\Delta}\bar{\delta}(x)$. Therefore, by Theorem 1.5(ii), $\mathbf{Alg}(\mathcal{L}) \models \bar{\varepsilon}(x) = \bar{\delta}(x) \leftrightarrow \bar{\varepsilon}'(x) = \bar{\delta}'(x)$. ■

DEFINITION 2.6 Let t_1, t_2 be algebraic similarity types. A function $\iota : \text{dom}(t_1) \rightarrow \text{Trm}_{t_2}$ is called a *term-translation of t_1 into t_2* iff for any k -ary t_1 -type function symbol f , $\iota(f)$ is a k -ary term of type t_2 . A term-translation always induces a function $\hat{\iota} : \text{Trm}_{t_1} \rightarrow \text{Trm}_{t_2}$ and a function $\tilde{\iota} : \text{Fmla}_{t_1} \rightarrow \text{Fmla}_{t_2}$ as follows:

- for any variable $x \in X$, $\hat{\iota}(x) \stackrel{\text{def}}{=} x$;
- if f is a k -ary function symbol of type t_1 and $\tau_0, \dots, \tau_{k-1} \in \text{Trm}_{t_1}$ then

$$\hat{\iota}(f(\tau_0, \dots, \tau_{k-1})) \stackrel{\text{def}}{=} \iota(f)(x_0/\hat{\iota}(\tau_0), \dots, x_{k-1}/\hat{\iota}(\tau_{k-1}));$$

- for any $\tau_0, \tau_1 \in \text{Trm}_{t_1}$, $\tilde{\iota}(\tau_0 = \tau_1) \stackrel{\text{def}}{=} (\hat{\iota}(\tau_0) = \hat{\iota}(\tau_1))$;
- for any $\varphi, \psi \in \text{Fmla}_{t_1}$,

$$\tilde{\iota}(\neg\varphi) \stackrel{\text{def}}{=} \neg\tilde{\iota}(\varphi), \quad \tilde{\iota}(\varphi \vee \psi) \stackrel{\text{def}}{=} \tilde{\iota}(\varphi) \vee \tilde{\iota}(\psi), \quad \tilde{\iota}(\exists x\varphi) \stackrel{\text{def}}{=} \exists x \tilde{\iota}(\varphi).$$

Similarly, the functions $\hat{\iota}$ and $\tilde{\iota}$ can be extended to sets of terms and formulas, respectively, by stipulating that for $\bar{\tau} \subseteq \text{Trm}_{t_1}$, $\hat{\iota}(\bar{\tau}) \stackrel{\text{def}}{=} \{\hat{\iota}(\tau) : \tau \in \bar{\tau}\}$ and for $\Gamma \subseteq \text{Fmla}_{t_1}$, $\tilde{\iota}(\Gamma) \stackrel{\text{def}}{=} \{\tilde{\iota}(\varphi) : \varphi \in \Gamma\}$. \square

Remark 2.7 A logic-translation I of some logic $\langle \text{Cn}(\mathcal{L}_1), \approx_{\mathcal{L}_1} \rangle$ into some logic $\langle \text{Cn}(\mathcal{L}_2), \approx_{\mathcal{L}_2} \rangle$ is in fact a term-translation of similarity type $\text{Cn}(\mathcal{L}_1)$ into $\text{Cn}(\mathcal{L}_2)$. Moreover, since formulas of \mathcal{L}_i ($i = 1, 2$) can be considered as $\text{Cn}(\mathcal{L}_i)$ -type terms, the function \hat{I} induced by I as a logic-translation is the same as \hat{I} induced by I as a term-translation. \square

Lemma 2.8 *If ι is a term-translation of t_1 into t_2 then for any $\Gamma \cup \{\varphi\} \subseteq \text{Fmla}_{t_1}$,*

$$\Gamma \models \varphi \implies \tilde{\iota}(\Gamma) \models \tilde{\iota}(\varphi).$$

Proof. It is easy to check that $\tilde{\iota}$ “preserves” the axioms and rules of any calculus for first-order logic. ■

DEFINITION 2.9

- (i) For $n = 1, 2$, let t_n be an algebraic similarity type, let \mathbf{K}_n be a t_n -type quasivariety and let $\langle \bar{\varepsilon}_n, \bar{\delta}_n, \bar{\Delta}_n \rangle$ be a deductivizator of \mathbf{K}_n . Let $\mathcal{A}_n \stackrel{\text{def}}{=} \langle t_n, \mathbf{K}_n, [\bar{\varepsilon}_n, \bar{\delta}_n, \bar{\Delta}_n]_{\mathbf{K}_n} \rangle$ ($n = 1, 2$). A term-translation ι from t_1 into t_2 is called an $(\mathcal{A}_1, \mathcal{A}_2)$ -interpretation iff

- (a) for any $\varphi \in Fml_{t_1}$,

$$\mathbf{K}_1 \models \varphi \implies \mathbf{K}_2 \models \tilde{\iota}(\varphi);$$

- (b)

$$\langle \hat{\iota}(\bar{\varepsilon}_1), \hat{\iota}(\bar{\delta}_1), \hat{\iota}(\bar{\Delta}_1) \rangle \simeq_{\mathbf{K}_2} \langle \bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2 \rangle.$$

We note that this definition is sensible because, by (i)(a), $\langle \hat{\iota}(\bar{\varepsilon}_1), \hat{\iota}(\bar{\delta}_1), \hat{\iota}(\bar{\Delta}_1) \rangle$ is a deductivizator of \mathbf{K}_2 .

- (ii) We define an equivalence relation on $(\mathcal{A}_1, \mathcal{A}_2)$ -interpretations as follows:

$$\iota \approx j \stackrel{\text{def}}{\iff} \text{ for any } \tau \in T_{rm_{t_1}}, \quad \mathbf{K}_2 \models \hat{\iota}(\tau) = \hat{j}(\tau).$$

Let $[[\iota]]$ denote the \approx -equivalence class of ι .

- (iii) Let $\mathcal{A} \stackrel{\text{def}}{=} \langle t, \mathbf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}} \rangle$ as above. Let $id_{\mathcal{A}}$ be the term-translation of t into t defined by $id_{\mathcal{A}}(f) \stackrel{\text{def}}{=} f(x_0, \dots, x_{k-1})$, for each k -ary function symbol $f \in \text{dom}(t)$. \square

We note that the function $\tilde{\iota}$ induced by an $(\mathcal{A}_1, \mathcal{A}_2)$ -interpretation is a special case of the well-investigated notion of “interpretation between first-order theories”, cf. e.g. Monk [11], Andr  ka–Gergely–N  meti [1], van Benthem–Pearce [15], Gergely [8], N  meti [12], [13].

The following lemma is an easy consequence of basic properties of equational logic.

Lemma 2.10

- (i) Let ι, ι' be $(\mathcal{A}_1, \mathcal{A}_2)$ -interpretations and let j, j' be $(\mathcal{A}_2, \mathcal{A}_3)$ -interpretations such that $\iota \approx \iota'$ and $j \approx j'$ hold. Then $\hat{j} \circ \iota$ and $\hat{j}' \circ \iota'$ are $(\mathcal{A}_1, \mathcal{A}_3)$ -interpretations, and $\hat{j} \circ \iota \approx \hat{j}' \circ \iota'$.
- (ii) $id_{\mathcal{A}}$ is an $(\mathcal{A}, \mathcal{A})$ -interpretation, and for any $(\mathcal{A}, \mathcal{A}')$ -interpretation ι , $\hat{\iota} \circ id_{\mathcal{A}} \approx \iota \approx \hat{id}_{\mathcal{A}'} \circ \iota$. ■

DEFINITION 2.11 The category **QVAR** of logic-generated quasivarieties is defined as follows.

$$\begin{aligned} \text{Obj}_{\mathbf{QVAR}} &\stackrel{\text{def}}{=} \{ \mathcal{A} : \mathcal{A} = \langle t, \mathbf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}} \rangle, t \text{ is an algebraic similarity type,} \\ &\quad \mathbf{K} \text{ is a } t\text{-type quasivariety, and} \\ &\quad \langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \text{ is a deductivizator of } \mathbf{K} \} \\ \text{Mor}_{\mathbf{QVAR}}(\mathcal{A}_1, \mathcal{A}_2) &\stackrel{\text{def}}{=} \{ [[\iota]] : \iota \text{ is an } (\mathcal{A}_1, \mathcal{A}_2)\text{-interpretation} \}, \\ &\quad \text{for any } \mathcal{A}_1, \mathcal{A}_2 \in \text{Obj}_{\mathbf{QVAR}} \\ ID_{\mathcal{A}} &\stackrel{\text{def}}{=} [[id_{\mathcal{A}}]], \quad \text{for any } \mathcal{A} \in \text{Obj}_{\mathbf{QVAR}} \\ [[j]][[\iota]] &\stackrel{\text{def}}{=} [[\hat{j} \circ \iota]], \quad \text{for any } \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \in \text{Obj}_{\mathbf{QVAR}}, \\ &\quad [[\iota]] \in \text{Mor}_{\mathbf{QVAR}}(\mathcal{A}_1, \mathcal{A}_2), [[j]] \in \text{Mor}_{\mathbf{QVAR}}(\mathcal{A}_2, \mathcal{A}_3). \end{aligned}$$

Then, by Lemma 2.10, **QVAR** is indeed a category. \square

3 Isomorphism

THEOREM 3.1 *ALOG and QVAR are isomorphic categories.*

Proof. To prove the theorem, we define functors $F_1 : \text{ALOG} \rightarrow \text{QVAR}$ and $F_2 : \text{QVAR} \rightarrow \text{ALOG}$, and prove that (i–iv) below hold.

- (i) for any $\mathcal{L} \in \text{Obj}_{\text{ALOG}}$, $F_2(F_1(\mathcal{L})) = \mathcal{L}$;
- (ii) for any $\mathcal{A} \in \text{Obj}_{\text{QVAR}}$, $F_1(F_2(\mathcal{A})) = \mathcal{A}$;
- (iii) for any $\mathcal{L}_1, \mathcal{L}_2 \in \text{Obj}_{\text{ALOG}}$, $[I] \in \text{Mor}_{\text{ALOG}}(\mathcal{L}_1, \mathcal{L}_2)$, $F_2(F_1([I])) = [I]$;
- (iv) for any $\mathcal{A}_1, \mathcal{A}_2 \in \text{Obj}_{\text{QVAR}}$, $[[i]] \in \text{Mor}_{\text{QVAR}}(\mathcal{A}_1, \mathcal{A}_2)$, $F_1(F_2([[i]])) = [[i]]$.

Step 1. The definition of functors F_1, F_2 on objects.

First, let \mathcal{L} be an algebraizable logical system and let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ be an algebraizator for \mathcal{L} . Then let

$$F_1(\mathcal{L}) \stackrel{\text{def}}{=} \langle \text{Cn}(\mathcal{L}), \text{Alg}(\mathcal{L}), [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\text{Alg}(\mathcal{L})} \rangle.$$

Note that this definition is sensible by Prop. 2.5.

Second, to define functor F_2 , let $\mathcal{A} \in \text{Obj}_{\text{QVAR}}$, $\mathcal{A} = \langle t, \mathbf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}} \rangle$. Then

$$F_2(\mathcal{A}) \stackrel{\text{def}}{=} \langle t, \wr_{F_2(\mathcal{A})} \rangle,$$

where for any $\Gamma \cup \{\varphi\} \subseteq \text{Trm}_t$,

$$\begin{aligned} \Gamma \wr_{F_2(\mathcal{A})} \varphi &\stackrel{\text{def}}{\iff} \text{there is some finite } \Gamma' \subseteq \Gamma \text{ such that} \\ &\mathbf{K} \models \bigwedge_{\psi \in \Gamma'} \bar{\varepsilon}(\psi) = \bar{\delta}(\psi) \rightarrow \bar{\varepsilon}(\varphi) = \bar{\delta}(\varphi). \end{aligned}$$

By Def. 2.4(ii), this definition is independent from the choice of representative $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ from the class $[\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}}$.

We show that $F_2(\mathcal{A})$ is an algebraizable logical system and

$$\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \text{ is an algebraizator for } F_2(\mathcal{A}). \quad (1)$$

Indeed, conditions (1–5) of Def. 1.1 hold for $F_2(\mathcal{A})$ by some basic properties of first-order logic. Since $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ is a deductivizator of \mathbf{K} , condition (6) of Def. 1.1 holds for $F_2(\mathcal{A})$ and $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ because of basic properties of equational logic.

Step 2. The proofs of statements (i–ii).

For (i): We show that for any algebraizable logical system $\mathcal{L} = \langle \text{Cn}(\mathcal{L}), \wr_{\mathcal{L}} \rangle$, $F_2(F_1(\mathcal{L})) = \mathcal{L}$ holds. Let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ be an algebraizator for \mathcal{L} , and let $F_2(F_1(\mathcal{L})) \stackrel{\text{def}}{=} \langle \text{Cn}(\mathcal{L}), \wr' \rangle$. Then for any $\Gamma \cup \{\varphi\} \subseteq \text{Fm}(\mathcal{L})$,

$$\begin{aligned} \Gamma \wr' \varphi &\iff (\text{by defs. of } F_1, F_2) \\ (\exists \Gamma' \subseteq \Gamma, \Gamma' \text{ is finite}) \text{Alg}(\mathcal{L}) \models \bigwedge_{\psi \in \Gamma'} \bar{\varepsilon}(\psi) = \bar{\delta}(\psi) &\rightarrow \bar{\varepsilon}(\varphi) = \bar{\delta}(\varphi) \iff (\text{by Thm. 1.5(i)}) \\ (\exists \Gamma' \subseteq \Gamma, \Gamma' \text{ is finite}) \Gamma' \wr_{\mathcal{L}} \varphi &\iff (\text{by Def. 1.1(2),(4)}) \\ \Gamma \wr_{\mathcal{L}} \varphi. \end{aligned}$$

For (ii): Let $\mathcal{A} = \langle t, \mathbf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}} \rangle$. We show that $F_1(F_2(\mathcal{A})) = \mathcal{A}$. By (1) above, it is enough to show that $\mathbf{K} = \mathbf{Alg}(F_2(\mathcal{A}))$ holds. To this end, let q be an arbitrary t -type quasi-equation of form $\tau_1 = \tau'_1 \wedge \dots \wedge \tau_k = \tau'_k \rightarrow \tau_0 = \tau'_0$. Then, by Theorem 1.5(ii),

$$\begin{aligned} \mathbf{Alg}(F_2(\mathcal{A})) \models q &\iff \{\tau_1 \bar{\Delta} \tau'_1, \dots, \tau_k \bar{\Delta} \tau'_k\} \wr_{F_2(\mathcal{A})} \tau_0 \bar{\Delta} \tau'_0 \\ &\stackrel{\text{def. of } F_2}{\iff} \mathbf{K} \models \bigwedge_{1 \leq i \leq k} \bar{\varepsilon}(\tau_i \bar{\Delta} \tau'_i) = \bar{\delta}(\tau_i \bar{\Delta} \tau'_i) \rightarrow \bar{\varepsilon}(\tau_0 \bar{\Delta} \tau'_0) = \bar{\delta}(\tau_0 \bar{\Delta} \tau'_0) \\ &\iff \mathbf{K} \models q, \end{aligned}$$

since $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ is a deductivizator of \mathbf{K} .

Step 3. The definition of functors F_1, F_2 on morphisms.

First, for any $(\mathcal{L}_1, \mathcal{L}_2)$ -interpretation I , let

$$F_1([I]) \stackrel{\text{def}}{=} [[I]].$$

We have to show that this definition is sensible that is,

- (a) if I is an $(\mathcal{L}_1, \mathcal{L}_2)$ -interpretation then I is also an $(F_1(\mathcal{L}_1), F_1(\mathcal{L}_2))$ -interpretation;
- (b) for any $(\mathcal{L}_1, \mathcal{L}_2)$ -interpretations I, J , if $I \sim J$ then also $I \approx J$.

Let $\langle \bar{\varepsilon}_j, \bar{\delta}_j, \bar{\Delta}_j \rangle$ be an algebraizator for \mathcal{L}_j ($j = 1, 2$).

For (a): First, we have to show that for any $\varphi \in Fml_{Cn(\mathcal{L}_1)}$, “ $\mathbf{Alg}(\mathcal{L}_1) \models \varphi \Rightarrow \mathbf{Alg}(\mathcal{L}_2) \models \tilde{I}(\varphi)$ ” holds. By Lemma 2.8, it is enough to prove this statement for quasi-equations, since $\mathbf{Alg}(\mathcal{L}_1) \models \varphi$ implies that there is some set Γ of quasi-equations such that $\mathbf{Alg}(\mathcal{L}_1) \models \Gamma$ and $\Gamma \models \varphi$ hold. Thus, assume that $\mathbf{Alg}(\mathcal{L}_1) \models (\tau_1 = \tau'_1 \wedge \dots \wedge \tau_k = \tau'_k \rightarrow \tau_0 = \tau'_0)$. Then, by Theorem 1.5(ii),

$$\begin{aligned} \{\tau_1 \bar{\Delta}_1 \tau'_1, \dots, \tau_k \bar{\Delta}_1 \tau'_k\} \wr_{\mathcal{L}_1} \tau_0 \bar{\Delta}_1 \tau'_0 &\implies \\ \{\hat{I}(\tau_1 \bar{\Delta}_1 \tau'_1), \dots, \hat{I}(\tau_k \bar{\Delta}_1 \tau'_k)\} \wr_{\mathcal{L}_2} \hat{I}(\tau_0 \bar{\Delta}_1 \tau'_0) &\iff \\ \{\hat{I}(\tau_1) \hat{I}(\bar{\Delta}_1) \hat{I}(\tau'_1), \dots, \hat{I}(\tau_k) \hat{I}(\bar{\Delta}_1) \hat{I}(\tau'_k)\} \wr_{\mathcal{L}_2} \hat{I}(\tau_0) \hat{I}(\bar{\Delta}_1) \hat{I}(\tau'_0) &\iff \text{(by Prop. 1.4)} \\ \{\hat{I}(\tau_1) \bar{\Delta}_2 \hat{I}(\tau'_1), \dots, \hat{I}(\tau_k) \bar{\Delta}_2 \hat{I}(\tau'_k)\} \wr_{\mathcal{L}_2} \hat{I}(\tau_0) \bar{\Delta}_2 \hat{I}(\tau'_0) &\iff \text{(by Theorem 1.5(ii))} \\ \mathbf{Alg}(\mathcal{L}_2) \models \tilde{I}(\tau_1 = \tau'_1 \wedge \dots \wedge \tau_k = \tau'_k \rightarrow \tau_0 = \tau'_0). \end{aligned}$$

Second, by Def. 2.1(ii)(b), $\langle \hat{I}(\bar{\varepsilon}_1), \hat{I}(\bar{\delta}_1), \hat{I}(\bar{\Delta}_1) \rangle$ is an algebraizator for \mathcal{L}_2 . Therefore, by Prop. 2.5,

$$\langle \hat{I}(\bar{\varepsilon}_1), \hat{I}(\bar{\delta}_1), \hat{I}(\bar{\Delta}_1) \rangle \simeq_{\mathbf{Alg}(\mathcal{L}_2)} \langle \bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2 \rangle$$

holds, as needed.

For (b): Assume $I \sim J$. Then $\wr_{\mathcal{L}_2} \hat{I}(\tau) \bar{\Delta}_2 \hat{J}(\tau)$, for any $\tau \in Fm(\mathcal{L}_1) = Trm_{Cn(\mathcal{L}_1)}$. Then, by Theorem 1.5(ii), $\mathbf{Alg}(\mathcal{L}_2) \models \hat{I}(\tau) = \hat{J}(\tau)$ holds, proving $I \approx J$.

Next, let $\mathcal{A}_1, \mathcal{A}_2 \in Obj_{\mathbf{QVAR}}$, $\mathcal{A}_k = \langle t_k, \mathbf{K}_k, [\bar{\varepsilon}_k, \bar{\delta}_k, \bar{\Delta}_k]_{\mathbf{K}_k} \rangle$ ($k = 1, 2$). For any $(\mathcal{A}_1, \mathcal{A}_2)$ -interpretation ι , let

$$F_2([\iota]) \stackrel{\text{def}}{=} [\iota].$$

We have to show that this definition is sensible that is,

- (c) if ι is an $(\mathcal{A}_1, \mathcal{A}_2)$ -interpretation then ι is also an $(F_2(\mathcal{A}_1), F_2(\mathcal{A}_2))$ -interpretation;
- (d) for any $(\mathcal{A}_1, \mathcal{A}_2)$ -interpretations ι, j , if $\iota \approx j$ then also $\iota \sim j$.

For (c): First, by Remark 2.7, we have to show that for any $\Gamma \cup \{\varphi\} \subseteq Fm(F_2(\mathcal{A}_1)) = Trm_{t_1}$, “ $\Gamma \approx_{F_2(\mathcal{A}_1)} \varphi \implies \hat{i}(\Gamma) \approx_{F_2(\mathcal{A}_2)} \hat{i}(\varphi)$ ” holds. Now assume that $\Gamma \approx_{F_2(\mathcal{A}_1)} \varphi$. Then, by definition, there is some finite $\Gamma' \subseteq \Gamma$ such that

$$\begin{aligned}
& K_1 \models \bigwedge_{\psi \in \Gamma'} \bar{\varepsilon}_1(\psi) = \bar{\delta}_1(\psi) \rightarrow \bar{\varepsilon}_1(\varphi) = \bar{\delta}_1(\varphi) \\
\implies & K_2 \models \tilde{i}(\bigwedge_{\psi \in \Gamma'} \bar{\varepsilon}_1(\psi) = \bar{\delta}_1(\psi) \rightarrow \bar{\varepsilon}_1(\varphi) = \bar{\delta}_1(\varphi)) \\
\iff & K_2 \models \bigwedge_{\psi \in \Gamma'} \hat{i}(\bar{\varepsilon}_1(\psi)) = \hat{i}(\bar{\delta}_1(\psi)) \rightarrow \hat{i}(\bar{\varepsilon}_1(\varphi)) = \hat{i}(\bar{\delta}_1(\varphi)) \\
\iff & K_2 \models \bigwedge_{\psi \in \Gamma'} \hat{i}(\bar{\varepsilon}_1)(\hat{i}(\psi)) = \hat{i}(\bar{\delta}_1)(\hat{i}(\psi)) \rightarrow \hat{i}(\bar{\varepsilon}_1)(\hat{i}(\varphi)) = \hat{i}(\bar{\delta}_1)(\hat{i}(\varphi)) \\
\iff & K_2 \models \bigwedge_{\psi \in \Gamma'} \bar{\varepsilon}_2(\hat{i}(\psi)) = \bar{\delta}_2(\hat{i}(\psi)) \rightarrow \bar{\varepsilon}_2(\hat{i}(\varphi)) = \bar{\delta}_2(\hat{i}(\varphi)) \\
\iff & \hat{i}(\Gamma) \approx_{F_2(\mathcal{A}_2)} \hat{i}(\varphi).
\end{aligned}$$

Second, let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ be an arbitrary algebraizator for $F_2(\mathcal{A}_1)$. We have to show that $\langle \hat{i}(\bar{\varepsilon}), \hat{i}(\bar{\delta}), \hat{i}(\bar{\Delta}) \rangle$ is an algebraizator for $F_2(\mathcal{A}_2)$. By (1) above, $\langle \bar{\varepsilon}_1, \bar{\delta}_1, \bar{\Delta}_1 \rangle$ is also an algebraizator for $F_2(\mathcal{A}_1)$, thus, by Prop. 2.5, $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ and $\langle \bar{\varepsilon}_1, \bar{\delta}_1, \bar{\Delta}_1 \rangle$ are both deductivizators of $\mathbf{Alg}(F_2(\mathcal{A}_1))$ with

$$\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \simeq_{\mathbf{Alg}(F_2(\mathcal{A}_1))} \langle \bar{\varepsilon}_1, \bar{\delta}_1, \bar{\Delta}_1 \rangle.$$

By statement (ii) above, $\mathbf{Alg}(F_2(\mathcal{A}_1)) = K_1$, thus $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \simeq_{K_1} \langle \bar{\varepsilon}_1, \bar{\delta}_1, \bar{\Delta}_1 \rangle$ holds. Since ι is an $(\mathcal{A}_1, \mathcal{A}_2)$ -interpretation, this implies that $\langle \hat{i}(\bar{\varepsilon}), \hat{i}(\bar{\delta}), \hat{i}(\bar{\Delta}) \rangle$ and $\langle \hat{i}(\bar{\varepsilon}_1), \hat{i}(\bar{\delta}_1), \hat{i}(\bar{\Delta}_1) \rangle$ are both deductivizators of K_2 and $\langle \hat{i}(\bar{\varepsilon}), \hat{i}(\bar{\delta}), \hat{i}(\bar{\Delta}) \rangle \simeq_{K_2} \langle \hat{i}(\bar{\varepsilon}_1), \hat{i}(\bar{\delta}_1), \hat{i}(\bar{\Delta}_1) \rangle$. Now, by (1) again, it follows that $\langle \hat{i}(\bar{\varepsilon}), \hat{i}(\bar{\delta}), \hat{i}(\bar{\Delta}) \rangle$ is an algebraizator for $F_2(\mathcal{A}_2)$.

For (d): Assume $\iota \approx j$, and let $\varphi \in Trm_{t_1} = Fm(F_2(\mathcal{A}_1))$. Then $K_2 \models \hat{i}(\varphi) = \hat{j}(\varphi)$ holds. Thus, by $\langle \bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2 \rangle$ being a deductivizator, $K_2 \models \bar{\varepsilon}_2(\hat{i}(\varphi)\bar{\Delta}_2\hat{j}(\varphi)) = \bar{\delta}_2(\hat{i}(\varphi)\bar{\Delta}_2\hat{j}(\varphi))$ follows. Then, by the definition of F_2 , $\approx_{F_2(\mathcal{A}_2)} \hat{i}(\varphi)\bar{\Delta}_2\hat{j}(\varphi)$, proving $\iota \sim j$.

The proofs of statements (iii) and (iv) above are immediate from the definitions of F_1 and F_2 .

We have proved that **ALOG** and **QVAR** are isomorphic categories. ■

4 Cocompleteness

THEOREM 4.1 *QVAR is a small-cocomplete category (i.e., all small colimits exist in it).*

The proof uses the following lemma.

Lemma 4.2 (cf. e.g. MacLane [10], p.109)

If a category has all coequalizers and all small coproducts then it is small-cocomplete.

Proof of Lemma 4.2. Here we give the sketch of the proof in order to illustrate that colimits in general are indeed “computable” if coequalizers and coproducts are given.

Let a small diagram \mathcal{D} be given. Let $\langle O_1, i_A \rangle_{A \in \text{Obj}_{\mathcal{D}}}$ be the coproduct cocone of all the objects of \mathcal{D} . Let \mathcal{M} denote the set of those objects of \mathcal{D} which are domains of some morphisms of \mathcal{D} , and let $\langle O_2, j_A \rangle_{A \in \mathcal{M}}$ be the coproduct of \mathcal{M} . Then the two cocones $\langle O_1, i_A \rangle_{A \in \mathcal{M}}$ and $\langle O_1, i_{Bm} \rangle_{A \in \mathcal{M}, B \in \text{Obj}_{\mathcal{D}}, m \in \text{Mor}_{\mathcal{D}}(A, B)}$ induce two morphisms f and g from O_2 to O_1 .

$$\begin{aligned}
& (\exists! f)(\forall A \in \mathcal{M}) f j_A = i_A \\
& (\exists! g)(\forall A \in \mathcal{M})(\forall B \in \text{Obj}_{\mathcal{D}})(\forall m \in \text{Mor}_{\mathcal{D}}(A, B)) g j_A = i_{Bm}
\end{aligned}$$

$$\begin{array}{ccc}
& & f \\
O_2 & \xrightarrow{\quad} & O_1 \\
& \nearrow g & \\
j_A \uparrow & & \uparrow i_B \\
A & \xrightarrow{\quad m \quad} & B \\
& \nwarrow i_A &
\end{array}$$

It is proved in MacLane [10] that the coequalizer of diagram $\langle O_1, O_2, f, g \rangle$ equals to the colimit of diagram \mathcal{D} . \blacksquare

Proof of Theorem 4.1. We give the small coproducts and the coequalizers in category **QVAR**.

Let \mathcal{D} be a small diagram in **QVAR** with $Obj_{\mathcal{D}} = \{\mathcal{A}_s : s \in S\} = \{\langle t_s, \mathbf{K}_s, [\bar{\varepsilon}_s, \bar{\delta}_s, \bar{\Delta}_s]_{\mathbf{K}_s} \rangle : s \in S\}$, for some set S , and having no morphisms. For each $s \in S$, let $Ax_s \subseteq Fmla_{t_s}$ be a set of t_s -type quasi-equations such that $Mod_{t_s}(Ax_s) = \mathbf{K}_s$. Let

$$\begin{aligned}
t &\stackrel{\text{def}}{=} \bigsqcup_{s \in S} t_s & (\bigsqcup \text{ denotes disjoint union}) \\
Ax &\stackrel{\text{def}}{=} \bigsqcup_{s \in S} Ax_s \cup \{(\bar{\varepsilon}_{s_1}(x) = \bar{\delta}_{s_1}(x)) \leftrightarrow (\bar{\varepsilon}_{s_2}(x) = \bar{\delta}_{s_2}(x)) : s_1, s_2 \in S\} \\
\mathbf{K} &\stackrel{\text{def}}{=} Mod_t(Ax).
\end{aligned}$$

Then for any $s_1, s_2 \in S$, $\langle \bar{\varepsilon}_{s_1}, \bar{\delta}_{s_1}, \bar{\Delta}_{s_1} \rangle \simeq_{\mathbf{K}} \langle \bar{\varepsilon}_{s_2}, \bar{\delta}_{s_2}, \bar{\Delta}_{s_2} \rangle$. Now let $s \in S$ be arbitrary and let

$$[\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}} \stackrel{\text{def}}{=} [\bar{\varepsilon}_s, \bar{\delta}_s, \bar{\Delta}_s]_{\mathbf{K}}.$$

Claim 4.2.1 $\langle \langle t, \mathbf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}} \rangle, [[id_{\mathcal{A}_s}]] \rangle_{s \in S}$ is the coproduct of \mathcal{D} .

Proof of Claim 4.2.1. Let $\mathcal{A} \stackrel{\text{def}}{=} \langle t, \mathbf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}} \rangle$ and $\mathcal{A}' \stackrel{\text{def}}{=} \langle t', \mathbf{K}', [\bar{\varepsilon}', \bar{\delta}', \bar{\Delta}']_{\mathbf{K}'} \rangle$. Assume that $\langle \mathcal{A}', [[j_s]] \rangle_{s \in S}$ is a cocone of \mathcal{D} . We have to prove that there is a unique $H \in Mor_{\mathbf{QVAR}}(\mathcal{A}, \mathcal{A}')$ such that $(\forall s \in S) H[[id_{\mathcal{A}_s}]] = [[j_s]]$.

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\exists! H} & \mathcal{A}' \\
[[id_{\mathcal{A}_s}]] \uparrow & & \nearrow [[j_s]] \\
\mathcal{A}_s & &
\end{array}$$

To this end, let $h : dom(t) \rightarrow Trm_{t'}$ be the following function. For any $s \in S$, $f \in dom(t_s)$,

$$h(f) \stackrel{\text{def}}{=} j_s(f).$$

Then h is a term-translation of t into t' with $\hat{h} \circ id_{\mathcal{A}_s} = j_s$, for any $s \in S$. We prove that

- (a) h is an $(\mathcal{A}, \mathcal{A}')$ -interpretation;
- (b) for any $(\mathcal{A}, \mathcal{A}')$ -interpretation h' with $\hat{h}' \circ id_{\mathcal{A}_s} \approx j_s$ ($s \in S$), $h \approx h'$ holds.

For (a): Since j_s is an $(\mathcal{A}_s, \mathcal{A}')$ -interpretation, $\langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle \simeq_{\mathbf{K}'} \langle \hat{j}_s(\bar{\varepsilon}_s), \hat{j}_s(\bar{\delta}_s), \hat{j}_s(\bar{\Delta}_s) \rangle$ holds, for any $s \in S$. Therefore, for any $s_1, s_2 \in S$,

$$\begin{aligned}
\langle \hat{j}_{s_1}(\bar{\varepsilon}_{s_1}), \hat{j}_{s_1}(\bar{\delta}_{s_1}), \hat{j}_{s_1}(\bar{\Delta}_{s_1}) \rangle &\simeq_{\mathbf{K}'} \langle \hat{j}_{s_2}(\bar{\varepsilon}_{s_2}), \hat{j}_{s_2}(\bar{\delta}_{s_2}), \hat{j}_{s_2}(\bar{\Delta}_{s_2}) \rangle \text{ i.e.,} \\
\mathbf{K}' \models (\hat{j}_{s_1}(\bar{\varepsilon}_{s_1})(x) = \hat{j}_{s_1}(\bar{\delta}_{s_1})(x)) &\leftrightarrow (\hat{j}_{s_2}(\bar{\varepsilon}_{s_2})(x) = \hat{j}_{s_2}(\bar{\delta}_{s_2})(x)).
\end{aligned} \tag{2}$$

Now let $\varphi \in Fmla_t$ and assume $\mathbf{K} \models \varphi$. Then $Ax \models \varphi$ thus, by Lemma 2.8,

$$\tilde{h}(Ax) \models \tilde{h}(\varphi). \tag{3}$$

By definition,

$$\begin{aligned}\tilde{h}(Ax) &= \biguplus_{s \in S} \tilde{h}(\tilde{id}_{\mathcal{A}_s}(Ax_s)) \cup \\ &\cup \{(\hat{j}_{s_1}(\bar{\varepsilon}_{s_1})(x) = \hat{j}_{s_1}(\bar{\delta}_{s_1})(x)) \leftrightarrow (\hat{j}_{s_2}(\bar{\varepsilon}_{s_2})(x) = \hat{j}_{s_2}(\bar{\delta}_{s_2})(x)) : s_1, s_2 \in S\}.\end{aligned}$$

Now, since $(\forall s \in S) \tilde{h} \circ \tilde{id}_{\mathcal{A}_s} = \tilde{j}_s$ and j_s is an $(\mathcal{A}_s, \mathcal{A}')$ -interpretation, (2) implies that $K' \models \tilde{h}(Ax)$. Thus, by (3), $K' \models \tilde{h}(\varphi)$ follows, as needed.

For (b): Let h' be an $(\mathcal{A}, \mathcal{A}')$ -interpretation with $\hat{h}' \circ id_{\mathcal{A}_s} \approx j_s$ ($s \in S$). Then for any $s \in S$, $\tau_s \in Trm_{t_s}$,

$$K' \models (\hat{h}' \circ id_{\mathcal{A}_s})^\wedge(\tau_s) = \hat{j}_s(\tau_s).$$

In particular, for any k -ary $f \in dom(t_s)$,

$$K' \models \hat{h}'(f(x_0, \dots, x_{k-1})) = \hat{j}_s(f(x_0, \dots, x_{k-1})).$$

By the definition of h , for any $s \in S$, for any k -ary $f \in dom(t_s)$,

$$K' \models \hat{h}(f(x_0, \dots, x_{k-1})) = \hat{j}_s(f(x_0, \dots, x_{k-1}))$$

also holds. Now, by induction on the structure of t -type terms, it follows that for any $\tau \in Trm_t$,

$$K' \models \hat{h}'(\tau) = \hat{h}(\tau),$$

proving $h' \approx h$.

Thus, by (a) and (b), $H \stackrel{\text{def}}{=} [[h]]$ is the unique morphism with $H[[id_{\mathcal{A}_s}]] = [[j_s]]$ ($s \in S$), proving Claim 4.2.1. \blacksquare

Now let $\mathcal{A}_i = \langle t_i, K_i, [\bar{\varepsilon}_i, \bar{\delta}_i, \bar{\Delta}_i]_{K_i} \rangle$ ($i = 1, 2$) be two objects of \mathbf{QVAR} , and let $[[h]], [[g]] \in Mor_{\mathbf{QVAR}}(\mathcal{A}_1, \mathcal{A}_2)$. Consider the following diagram \mathcal{E} .

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{[[h]]} & \mathcal{A}_2 \\ & \searrow [[g]] & \end{array}$$

Let $Ax_2 \subseteq Fmla_{t_2}$ be a set of t_2 -type quasi-equations such that $Mod_{t_2}(Ax_2) = K_2$, and let

$$\begin{aligned}Ax &\stackrel{\text{def}}{=} Ax_2 \cup \{\hat{h}(f(x_0, \dots, x_{k-1})) = \hat{g}(f(x_0, \dots, x_{k-1})) : f \in dom(t_1) \text{ } k\text{-ary}\} \\ K &\stackrel{\text{def}}{=} Mod_{t_2}(Ax).\end{aligned}$$

Claim 4.2.2 $\langle \langle t_2, K, [\bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2]_{K} \rangle, [[id_{\mathcal{A}_2}]] \rangle$ is the colimit of \mathcal{E} .

Proof of Claim 4.2.2. First, it can be proved, by induction on the structure of t_1 -type terms, that for any $\tau \in Trm_{t_1}$, $K \models \hat{h}(\tau) = \hat{g}(\tau)$. Therefore, since $(id_{\mathcal{A}_2} \circ h)^\wedge = \hat{h}$ and $(id_{\mathcal{A}_2} \circ g)^\wedge = \hat{g}$, $[[id_{\mathcal{A}_2}]] [[h]] = [[id_{\mathcal{A}_2}]] [[g]]$ follows.

Second, let $\mathcal{A} \stackrel{\text{def}}{=} \langle t_2, K, [\bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2]_{K} \rangle$ and take an object $\mathcal{A}' \stackrel{\text{def}}{=} \langle t', K', [\bar{\varepsilon}', \bar{\delta}', \bar{\Delta}']_{K'} \rangle$ of \mathbf{QVAR} and some $[[j]] \in Mor_{\mathbf{QVAR}}(\mathcal{A}_2, \mathcal{A}')$ with $[[j]] [[h]] = [[j]] [[g]]$. We have to show that there is a unique $I \in Mor_{\mathbf{QVAR}}(\mathcal{A}, \mathcal{A}')$ such that $I [[id_{\mathcal{A}_2}]] = [[j]]$.

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{[[h]]} & \mathcal{A}_2 \\ & \searrow [[g]] & \downarrow [[j]] \\ \mathcal{A} & \xrightarrow{\exists! I} & \mathcal{A}' \end{array}$$

We show that $I \stackrel{\text{def}}{=} [[j]]$ is an appropriate choice that is,

- (c) j is an $(\mathcal{A}, \mathcal{A}')$ -interpretation;
- (d) for any $(\mathcal{A}, \mathcal{A}')$ -interpretation j' with $j' \circ id_{\mathcal{A}_2} \approx j$, $j' \approx j$ holds.

For (c): First, since j is an $(\mathcal{A}_2, \mathcal{A}')$ -interpretation,

$$\begin{aligned} \langle \hat{j}(\bar{\varepsilon}_2), \hat{j}(\bar{\delta}_2), \hat{j}(\bar{\Delta}_2) \rangle &\simeq_{\mathbf{K}'} \langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle \quad \text{and} \\ \mathbf{K}' &\models \hat{j}(Ax_2). \end{aligned} \tag{4}$$

Second, since $[[j]][[h]] = [[j]][[g]]$, thus for any k -ary function symbol of type t_1 ,

$$\begin{aligned} \mathbf{K}' &\models \hat{j}(\hat{h}(f(x_0, \dots, x_{k-1}))) = \hat{j}(\hat{g}(f(x_0, \dots, x_{k-1}))) \iff \\ \mathbf{K}' &\models \hat{j}(\hat{h}(f(x_0, \dots, x_{k-1}))) = \hat{g}(f(x_0, \dots, x_{k-1}))). \end{aligned} \tag{5}$$

Now let $\varphi \in Fml_{t_2}$ and assume $\mathbf{K} \models \varphi$. By Lemma 2.8, $\hat{j}(Ax) \models \hat{j}(\varphi)$ holds. Therefore, by (4) and (5), $\mathbf{K}' \models \hat{j}(\varphi)$ follows.

Item (d) can be proved analogously to item (b) in the proof of Claim 4.2.1 above. ■

We have proved that small coproducts and coequalizers exist in category QVAR. Now, by Lemma 4.2, all small colimits exist in QVAR. ■

Corollary 4.3 *ALOG is a small-cocomplete category.* ■

We note that though colimits always exist in ALOG, they are not always “interesting”. E.g. if \mathcal{L}_1 and \mathcal{L}_2 are two different algebraizable logical systems with $\text{Alg}(\mathcal{L}_1) = \text{Alg}(\mathcal{L}_2)$ then their coproduct in ALOG is an inconsistent logic.

The proof of Theorem 4.1 also yields the following result.

Corollary 4.4 *Let \mathcal{D} be a small diagram of QVAR, having objects $\langle t_s, \mathbf{K}_s, [\bar{\varepsilon}_s, \bar{\delta}_s, \bar{\Delta}_s]_{\mathbf{K}_s} \rangle$ for some set S , and having arbitrary morphisms. Let $\langle t, \mathbf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}} \rangle$ be the colimit of \mathcal{D} . If for each $s \in S$, \mathbf{K}_s is a finitely axiomatizable quasivariety then \mathbf{K} is also finitely axiomatizable.* ■

From the point of view of logics, this corollary means that any combination of *finitely axiomatizable logics*¹ is also finitely axiomatizable.

5 Discussion

In this paper only the first steps have been taken toward a systematic study of combining arbitrary logics by turning them to usual first order logic. Investigation can be extended to the study of categories of logics, where e.g. the consequence relation is *not compact* ((4) of Def. 1.1 is missing); or where condition (6)(v) of Def. 1.1 is missing (called *congruential logics* in Blok-Pigozzi [4]); or where condition (6) of Def. 1.1 is missing altogether (called *structural logics* in [4]).

An even more ambitious task is to develop the category theoretic “reconstruction” of combining logics which are given not merely with their consequence relations but also together with their semantics. (Algebraization of these kinds of logics is given e.g. Andr  ka et al [2], [3], [14].) This kind of “modelling” should be capable to reconstruct how the semantics of a combined logic is built up from the semantics of its “components”.

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¹“logics admitting finite Hilbert-style inference systems” in Andr  ka et al [2], or “finite deductive systems” in Blok-Pigozzi [4]

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