# ON MODAL LOGICS BETWEEN $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ AND $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$

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Abstract. We prove that every $n$-modal logic between $\mathbf{K}^n$ and $\mathbf{S5}^n$ is undecidable, whenever $n \geq 3$. We also show that each of these logics is non-finitely axiomatizable, lacks the product finite model property, and there is no algorithm deciding whether a finite frame validates the logic. These results answer several questions of Gabbay and Shehtman. The proofs combine the modal logic technique of Yankov–Fine frame formulas with algebraic logic results of Halmos, Johnson and Monk, and give a reduction of the (undecidable) representation problem of finite relation algebras.

## 1. Introduction and results

Here we deal with axiomatization and decision problems of *$n$-modal logics*: propositional multi-modal logics having finitely many unary modal operators $\Diamond_0, \ldots, \Diamond_{n-1}$ (and their duals $\Box_0, \ldots, \Box_{n-1}$), where $n$ is a non-zero natural number. Formulas of this language, using propositional variables from some fixed countably infinite set, are called *$n$-modal formulas*. Frames for $n$-modal logics — *$n$-frames* — are structures of the form $\mathcal{F} = (F, R_0, \ldots, R_{n-1})$ where $R_i$ is a binary relation on $F$, for each $i < n$. A *model* on an $n$-frame $\mathcal{F} = (F, R_0, \ldots, R_{n-1})$ is a pair $\mathfrak{M} = (\mathcal{F}, v)$ where $v$ is a function mapping the propositional variables into subsets of $F$. The inductive definition of "*formula $\varphi$ is true at point $x$ in model $\mathfrak{M}$*" is the standard one, e.g., the clause for $\Diamond_i$ ($i < n$) is as follows:

$$\mathfrak{M}, x \models \Diamond_i \psi \quad \text{iff} \quad \exists y \ (x R_i y \text{ and } \mathfrak{M}, y \models \psi).$$

Given an $n$-frame $\mathcal{F}$ and an $n$-modal formula $\varphi$, we say that $\varphi$ is *satisfiable* in $\mathcal{F}$ if $\mathfrak{M}, x \models \varphi$ for some model $\mathfrak{M}$ on $\mathcal{F}$ and point $x$ in $F$. Similarly, $\varphi$ is *valid* in $\mathcal{F}$ if $\mathfrak{M}, x \models \varphi$ for all such $\mathfrak{M}$ and $x$. $\mathcal{F}$ is a *frame for* a set $L$ of $n$-modal formulas if all formulas of $L$ are valid in $\mathcal{F}$. $L$ is called a *Kripke complete $n$-modal logic* if there is some class $\mathcal{C}$ of $n$-frames such that $L$ is the set of all $n$-modal formulas which are valid in every member of $\mathcal{C}$. This case we also say that $L$ is the *logic of* $\mathcal{C}$. Well-known Kripke complete unimodal logics are $\mathbf{K}$ (the logic of all 1-frames) and $\mathbf{S5}$ (the logic of all 1-frames $(W, R)$ with $R$ being an equivalence relation on $W$).

Special $n$-frames are the following (*$n$-ary*) *product frames.* Given 1-frames (i.e., usual Kripke frames for unimodal logic) $\mathcal{F}_0 = (W_0, R_0), \ldots, \mathcal{F}_{n-1} = (W_{n-1}, R_{n-1})$, their product $\mathcal{F}_0 \times \cdots \times \mathcal{F}_{n-1}$ is defined to be the relational structure

$$(W_0 \times \cdots \times W_{n-1}, \bar{R}_0, \ldots, \bar{R}_{n-1})$$

where, for each $i < n$, $\bar{R}_i$ is the following binary relation on $W_0 \times \cdots \times W_{n-1}$:

$$(u_0, \ldots, u_{n-1}) \bar{R}_i (v_0, \ldots, v_{n-1}) \quad \text{iff} \quad u_i R_i v_i \ \text{ and } \ u_k = v_k, \quad \text{for } k \neq i.$$

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For each $i < n$, let $L_i$ be a Kripke complete unimodal logic (of the language having modal operators $\Diamond_i$ and $\Box_i$). Define the *($n$-dimensional) product logic*

$$L_0 \times \cdots \times L_{n-1}$$

as the logic of the class of those product frames $(W_0, R_0) \times \cdots \times (W_{n-1}, R_{n-1})$ where, for each $i < n$, $(W_i, R_i)$ is a frame for $L_i$. For example, $\mathbf{K}^n$ is the logic of all $n$-ary product frames. It is not hard to see that $\mathbf{S5}^n$ is the logic of all $n$-ary products of *universal* 1-frames, that is, 1-frames $(W_i, R_i)$ with $R_i = W_i \times W_i$ $(i < n)$. Throughout, product frames of this kind are called *universal product* $\mathbf{S5}^n$*-frames*. We write $(W_0, \ldots, W_{n-1})$ for such a frame, and sometimes call it the *universal product frame on* $W_0 \times \cdots \times W_{n-1}$. Note that given a product frame

$$\mathcal{F} = (W_0 \times \cdots \times W_{n-1}, \bar{R}_0, \ldots, \bar{R}_{n-1}) = (W_0, R_0) \times \cdots \times (W_{n-1}, R_{n-1}),$$

for every $i < n$ the '$i$-reduct' $\mathcal{F}^{(i)} = (W_0 \times \cdots \times W_{n-1}, \bar{R}_i)$ of $\mathcal{F}$ is a union of disjoint copies of the 1-frame $(W_i, R_i)$. Thus the same modal formulas are valid in $\mathcal{F}^{(i)}$ and $(W_i, R_i)$. As a consequence we have that $L = L_0 \times \cdots \times L_{n-1}$ always includes $L_i$ $(i < n)$, and for every product frame $\mathcal{F} = \mathcal{F}_0 \times \cdots \times \mathcal{F}_{n-1}$,

$$\mathcal{F} \text{ is a frame for } L \quad \text{iff} \quad \mathcal{F}_i \text{ is a frame for } L_i, \text{ for all } i < n.$$

Products of modal logics have been studied in both pure modal logic (see Segerberg [16], Shehtman [17], Gabbay–Shehtman [5]) and in computer science applications (see Wolter–Zakharyaschev [19], [20], Gabbay *et al.* [3] and the references therein). Product logics are also relevant to finite variable fragments of modal and intermediate predicate logics, see Gabbay–Shehtman [4]. Axiomatization, decision and complexity problems of two-dimensional products were thoroughly investigated in [5], Marx [14], Spaan [18]. In higher dimensions — $n \geq 3$ from now on — the first results related to product logics were obtained in algebraic logic. This is due to the fact that the modal algebras corresponding to $\mathbf{S5}^n$ are well-known in this area: the *representable diagonal-free cylindric algebras of dimension $n$*. Thus the respective algebraic logic results of Johnson [9] and Maddux [12] imply that $\mathbf{S5}^n$ is non-finitely axiomatizable and undecidable.

Given a recursively enumerable set $L$ of $n$-modal formulas, if we can enumerate those formulas which are not in $L$ then we obtain a decision algorithm for $L$. Obviously, this can be done if

(A) $L$ has the *finite model property*, i.e., for every $n$-modal formula $\varphi$ which is not in $L$ there is a finite frame for $L$ where $\varphi$ is not valid; and
(B) finite frames for $L$ are recursively enumerable (up to isomorphism).

For instance, if $L$ is a finitely axiomatizable Kripke complete logic then (B) clearly holds for $L$. Product logics are defined as sets of modal formulas which are valid in classes of product frames. It is important to stress that in general there are other 'non-standard', i.e. non-product frames for such logics. Thus we can enumerate those formulas which are not in a product logic $L$ if

(C) $L$ has the *product finite model property*, i.e., for every $n$-modal formula $\varphi$ which is not in $L$ there is a finite product frame for $L$ where $\varphi$ is not valid; and
(D) finite product frames for $L$ are recursively enumerable (up to isomorphism).

Clearly, if $L$ is a product of finitely axiomatizable Kripke complete logics—such as e.g. $\mathbf{K}^n$, $\mathbf{K4}^n$, $\mathbf{S5}^n$—then (D) holds for $L$. Thus the undecidability and recursive enumerability of $\mathbf{S5}^n$ (see e.g. Henkin *et al*. [7]) imply that $\mathbf{S5}^n$ does not have the product finite model property. Further, (C) obviously implies (A). The reverse implication does not necessarily hold: $\mathbf{K}^n$ has the finite model property for every $n$ (Gabbay–Shehtman [5]), but lacks the product finite model property whenever $n \geq 3$ (see Theorem 4). Undecidability and the lack of product finite model property for all product logics between $\mathbf{K4}^n$ and $\mathbf{S5}^n$ was first proved by Zakharyaschev (see [3]). Non-finite axiomatizability of $\mathbf{K}^n$ was shown in Kurucz [10]. The fact that $\mathbf{K}^n$ has the finite model property while $\mathbf{S5}^n$ ($n \geq 3$) does not (Kurucz [11]) gave some hope about the decidability of $\mathbf{K}^n$. As our results below show, this is not the case: in higher dimensions all logics between $\mathbf{K}^n$ and $\mathbf{S5}^n$ are quite complicated.

Let $n \geq 3$ and let $L$ be any set of $n$-modal formulas with $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then the following hold.

**Theorem 1.** *$L$ is undecidable.*

**Theorem 2.** *It is undecidable whether a finite frame is a frame for $L$.*

**Theorem 3.** *$L$ is not finitely axiomatizable.*

**Theorem 4.** *$L$ does not have the product finite model property in the following strong sense: there is some* (3*-modal*) *formula $\varphi$ which does not belong to $L$ but $\varphi$ is valid in all finite $k$-ary product frames, for all $k \geq 3$.*

Theorems 4, 1 and 3 answer questions 20, 22 and 24 of Gabbay–Shehtman [5] (cf. also Q16.163 of Gabbay [2]): $\mathbf{K}^n$ lacks the product finite model property for $n \geq 3$, $\mathbf{K}^3$ is undecidable, and all the logics of the form $L \times \mathbf{S5}^2$ are undecidable and non-finitely axiomatizable, if $\mathbf{K} \subseteq L \subseteq \mathbf{S5}$. Thus $\mathbf{K}^3$ is a natural example of an undecidable but recursively enumerable logic which has the finite model property.

In the proofs we will use the following result of Hirsch–Hodkinson [8]:

($*$) *It is undecidable whether a finite simple relation algebra is representable.*[1]

For any natural number $n \geq 3$ and any finite simple relation algebra $\mathfrak{A}$ (see Section 3 for definitions), we define (in a recursive way) a finite $n$-frame $\mathcal{F}_{\mathfrak{A},n}$ and a 3-modal formula $\varphi_{\mathfrak{A}}$, and prove the following lemmas.

**Lemma 5.** *Let $L$ be any set of $n$-modal formulas with $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$, for some $n \geq 3$. Then the following are equivalent:*

(i) *$\mathcal{F}_{\mathfrak{A},n}$ is a frame for $L$.*
(ii) *The formula $\neg\varphi_{\mathfrak{A}}$ does not belong to $L$.*
(iii) *$\mathcal{F}_{\mathfrak{A},3}$ is a p-morphic image of some universal product $\mathbf{S5}^3$-frame.*

**Lemma 6.** *$\mathfrak{A}$ is representable iff $\mathcal{F}_{\mathfrak{A},3}$ is a p-morphic image of some universal product $\mathbf{S5}^3$-frame. Further, $\mathfrak{A}$ is representable with a finite base iff $\mathcal{F}_{\mathfrak{A},3}$ is a p-morphic image of some finite universal product $\mathbf{S5}^3$-frame.*

Now Theorems 1 and 2 follow straightforwardly from ($*$) and Lemmas 5, 6. Theorem 3 follows from Theorem 2, since if $L$ were finitely axiomatizable then there would be a recursive test for finite frames being frames for $L$. We prove Lemmas 5, 6 and Theorem 4 in Section 3. Note that if $L$ is recursively enumerable

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[1]In [8] this statement is not claimed for finite *simple* relation algebras, but for finite relation algebras in general only. However, this implies the result also for finite simple relation algebras, by taking subdirect decompositions. Or, in another way: the relation algebras constructed in the proof of [8] are clearly simple, thus the proof therein works for simple relation algebras as well.

and finite product frames for $L$ are also recursively enumerable (such as, e.g., for $\mathbf{K}^n$, $\mathbf{K4}^n$, $\mathbf{S5}^n$) then the lack of product finite model property for $L$ already follows from Theorem 1.

## 2. Frame formulas in product frames

In this section we establish a connection between arbitrary product frames and product frames for $\mathbf{S5}^3$. This connection (Claim 7 below) is the heart of the proof of Lemma 5.

Let $\mathcal{F} = (F, R_0, R_1, R_2)$ be a finite 3-frame with the following property:

$$(\forall p, p' \in F)(\exists s_0, s_1 \in F)\ pR_0s_0,\ s_0R_1s_1 \text{ and } s_1R_2p'. \tag{1}$$

(For example, universal product $\mathbf{S5}^3$-frames have this property.) For each point $p \in F$, introduce a propositional variable, denoted also by $p$. Define $\varphi_{\mathcal{F}}$ as the *Yankov–Fine frame formula of* $\mathcal{F}$:

$$\Box^+ \bigvee_{p\in F} (p \wedge \neg \bigvee_{p' \in F-\{p\}} p') \tag{2}$$

$$\wedge\ \Box^+ \bigwedge_{\substack{i<3, p,p'\in F\\ pR_ip'}} p \to \Diamond_i p' \tag{3}$$

$$\wedge\ \Box^+ \bigwedge_{\substack{i<3, p,p'\in F\\ \neg(pR_ip')}} p \to \neg\Diamond_i p'. \tag{4}$$

Here, $\Box_i^+\psi$ abbreviates $\psi \wedge \Box_i\psi$, and $\Box^+\psi$ abbreviates $\Box_0^+\Box_1^+\Box_2^+\psi$. Then clearly $\varphi_{\mathcal{F}}$ is satisfiable in $\mathcal{F}$: Take the model $\mathfrak{M} = (\mathcal{F}, \upsilon)$ with $\upsilon(p) = \{p\}$. Then $\mathfrak{M}, q \models \varphi_{\mathcal{F}}$, for any $q \in F$. Moreover, it is straightforward to see the following (cf. [1] for the unimodal case):

> *For any* 3*-frame* $\mathcal{H}$ *with property* (1)*,* $\mathcal{H}$ *satisfies* $\varphi_{\mathcal{F}}$ *iff there is a generated subframe* $\mathcal{H}^-$ *of* $\mathcal{H}$ *which maps p-morphically onto* $\mathcal{F}$*.*

The following claim is a modification of this statement which applies to arbitrary product frames satisfying $\varphi_{\mathcal{F}}$.

Claim 7. *Let* $\mathcal{F} = (F, R_0, R_1, R_2)$ *be a finite* 3*-frame such that the* $R_i$ *are equivalence relations and* (1) *holds in* $\mathcal{F}$*. If* $\varphi_{\mathcal{F}}$ *is satisfiable in an* $n$*-ary product frame* $\mathcal{H}$*, for some* $n \geq 3$*, then there is a universal product* $\mathbf{S5}^3$*-frame* $\mathcal{H}^-$ *which maps p-morphically onto* $\mathcal{F}$*. Further, if* $\mathcal{H}$ *is finite then* $\mathcal{H}^-$ *can be chosen finite as well.*

*Proof.* Assume $\varphi_{\mathcal{F}}$ is satisfiable in an $n$-ary product frame

$$\mathcal{H} = (U_0, S_0) \times (U_1, S_1) \times (U_2, S_2) \times \cdots \times (U_{n-1}, S_{n-1}).$$

Let the model $\mathfrak{M}$ on $\mathcal{H}$ and $u_i \in U_i$ $(i < n)$ be such that

$$\mathfrak{M}, (u_0, u_1, u_2, u_3, \ldots, u_{n-1}) \models \varphi_{\mathcal{F}}.$$

We fix $u_3, \ldots, u_{n-1}$ and write $v_0v_1v_2\bar{u}$ for points $(v_0, v_1, v_2, u_3, \ldots, u_{n-1})$ of $\mathcal{H}$. For $i < 3$, take

$$U_i^- = \{v \in U_i\ :\ v = u_i \text{ or } u_iS_iv\}.$$

Define a function $h : U_0^- \times U_1^- \times U_2^- \to F$ as follows:

$$h(v_0, v_1, v_2) = p \quad \text{iff} \quad \mathfrak{M}, v_0v_1v_2\bar{u} \models p.$$

Then $h$ is well-defined by (2). We claim that $h$ is a p-morphism from the universal product $\mathbf{S5}^3$-frame $\mathcal{H}^- = (U_0^-, U_1^-, U_2^-)$ onto $\mathcal{F}$.

First, $h$ is onto by (2), (3) and (1). Next we show that if $i < 3$, $(p_0, p_1, p_2)$, $(q_0, q_1, q_2) \in U_0^- \times U_1^- \times U_2^-$, $p_j = q_j$ for $j \neq i$, $j < 3$, $\mathfrak{M}, p_0p_1p_2\bar{u} \models p$ and $\mathfrak{M}, q_0q_1q_2\bar{u} \models p'$ then $pR_ip'$. We may assume without loss of generality that $i = 0$. By definition of $U_0^-$, either $p_0 = u_0$ or $u_0S_0p_0$, and similarly, either $q_0 = u_0$ or $u_0S_0q_0$. By (2), there is a unique $p'' \in F$ with $\mathfrak{M}, u_0p_1p_2\bar{u} \models p''$. We claim that $p''R_0p$ and $p''R_0p'$. Indeed, if $p_0 = u_0$ then $p = p''$, thus $p''R_0p$ holds by reflexivity of $R_0$. If $u_0S_0p_0$ then $\mathfrak{M}, u_0p_1p_2\bar{u} \models p'' \wedge \Diamond_0 p$ which, by (4), implies that $p''R_0p$. Similarly, one can show that $p''R_0p'$. Now $pR_0p'$ follows, by symmetry and transitivity of $R_0$.

Finally, we show that if $(p_0, p_1, p_2) \in U_0^- \times U_1^- \times U_2^-$, $\mathfrak{M}, p_0p_1p_2\bar{u} \models p$ and $pR_0p'$ then there is some $v \in U_0^-$ such that $\mathfrak{M}, vp_1p_2\bar{u} \models p'$. Similar statements hold for 1 and $U_1^-$, and 2 and $U_2^-$, respectively. Indeed, as we saw in the previous paragraph, $p''R_0p$ for the unique $p'' \in F$ with $\mathfrak{M}, u_0p_1p_2\bar{u} \models p''$. Then $p''R_0p'$ follows by transitivity of $R_0$. By (3), $\mathfrak{M}, u_0p_1p_2\bar{u} \models \Diamond_0 p'$ holds, thus there is some $v \in U_0^-$ with $\mathfrak{M}, vp_1p_2\bar{u} \models p'$.

Note that in general, even if $\mathcal{H}$ is a ternary product frame, $\mathcal{H}^-$ is far from being a subframe of $\mathcal{H}$. However, the set of points of $\mathcal{H}^-$ is in a one-to-one correspondence with a subset of the set of points of $\mathcal{H}$. So if $\mathcal{H}$ is finite then $\mathcal{H}^-$ is finite as well. □

## 3. Relation algebras and product frames

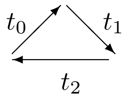
A *relation algebra* is a structure of form $\mathfrak{A} = (A, +, \cdot, -, 1, 0, ;, \breve{}, 1')$ satisfying the following properties, for all $x, y, z \in A$:

- $(A, +, \cdot, -, 1, 0)$ is a Boolean algebra
- $x ;(y ; z) = (x ; y) ; z$
- $x^{\breve{}\breve{}} = x$ and $x ; 1' = 1' ; x = x$
- ; and $\breve{}$ distribute over + (thus they are monotone with respect to Boolean $\leq$)
- *cycle law*: $x \cdot (y ; z) = 0 \iff y \cdot (x ; z^{\breve{}}) = 0 \iff z \cdot (y^{\breve{}} ; x) = 0.$

Note that this list of properties is not the "official" (equational) axiomatization for relation algebras: though it is equivalent, see Maddux [13] for a discussion. A relation algebra is *atomic* if its Boolean reduct is an atomic Boolean algebra. Thus, finite relation algebras are atomic. A relation algebra is *simple* if it has no non-trivial homomorphic images. It is well-known (cf. e.g., [13, Thm.17]) that a relation algebra $\mathfrak{A}$ is simple iff $1 ; a ; 1 = 1$ holds, for all $a \neq 0$ in $\mathfrak{A}$.

A natural example is the (simple) relation algebra of all subsets of $U \times U$, for some non-empty set $U$. Here ; is the composition (relative product) of binary relations, $\breve{}$ is converse (inverse), and $1'$ is the identity relation on $U$. A simple relation algebra is called *representable with base* $U$ if it is embeddable into the relation algebra of all subsets of $U \times U$. As we already mentioned, it follows from the main result of [8] that there is no algorithm deciding whether a finite simple relation algebra is representable.

Now take some finite simple relation algebra $\mathfrak{A}$. Call a triple $(t_0, t_1, t_2)$ of atoms of $\mathfrak{A}$ *consistent* if $t_2^{\breve{}} \leq t_0 ; t_1$ holds.

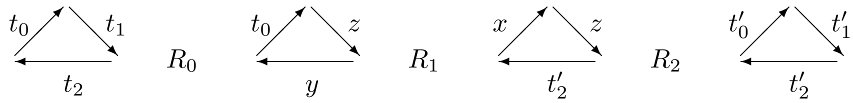



Note that, by the cycle law, if a triple $(t_0, t_1, t_2)$ is consistent then $(t_1, t_2, t_0)$, $(t_2, t_0, t_1)$, $(\breve{t_0}, \breve{t_2}, \breve{t_1})$, $(\breve{t_2}, \breve{t_1}, \breve{t_0})$ and $(\breve{t_1}, \breve{t_0}, \breve{t_2})$ are also consistent.

**Definition of the $n$-frame $\mathcal{F}_{\mathfrak{A},n}$ and the 3-modal formula $\varphi_{\mathfrak{A}}$:** Introduce a point $t = t_0 t_1 t_2$ for each consistent triple $(t_0, t_1, t_2)$ of atoms of $\mathfrak{A}$. Write $\mathcal{T}_{\mathfrak{A}}$ for the set of all such points. For $t, t' \in \mathcal{T}_{\mathfrak{A}}$ and $i < 3$ define $tR_i t'$ iff $t_i = t'_i$. For $3 \leq i < n$, let $R_i$ be the identity on $\mathcal{T}_{\mathfrak{A}}$, and let $\mathcal{F}_{\mathfrak{A},n} = (\mathcal{T}_{\mathfrak{A}}, R_0, R_1, R_2, \ldots, R_{n-1})$. Then clearly $\mathcal{F}_{\mathfrak{A},n}$ is finite and the $R_i$ are equivalence relations.

Claim 8. *$\mathcal{F}_{\mathfrak{A},3}$ has property* (1) *above.*

*Proof.* Take some $t, t' \in \mathcal{T}_{\mathfrak{A}}$. Since ; and $\breve{}$ are monotone and $\mathfrak{A}$ is simple, there are atoms $x, y$ of $\mathfrak{A}$ with $\breve{t_0} \leq x^{\smile} ; t_2'^{\smile} ; y$. Thus there is an atom $z$ such that $\breve{t_0} \leq z ; y$ and $z \leq x^{\smile} ; t_2'^{\smile}$. Now we have the following chain of consistent triples:




□

Now define $\varphi_{\mathfrak{A}}$ as the Yankov–Fine frame formula of $\mathcal{F}_{\mathfrak{A},3}$ (cf. Section 2).

*Proof of Lemma 5.* For (i) implies (ii): Assume $\mathcal{F}_{\mathfrak{A},n}$ is a frame for $L$. Since $\varphi_{\mathfrak{A}}$ is 3-modal and satisfiable in $\mathcal{F}_{\mathfrak{A},3}$, it is satisfiable in $\mathcal{F}_{\mathfrak{A},n}$, for any $n \geq 3$. Therefore, $\neg\varphi_{\mathfrak{A}}$ is not valid in $\mathcal{F}_{\mathfrak{A},n}$, thus it does not belong to $L$.

For (iii) implies (i): Suppose $\mathcal{F}_{\mathfrak{A},3}$ is a p-morphic image of some universal product $\mathbf{S5}^3$-frame $\mathcal{G}_0 \times \mathcal{G}_1 \times \mathcal{G}_2$. Then clearly $\mathcal{F}_{\mathfrak{A},n}$ is a p-morphic image of the universal product $\mathbf{S5}^n$-frame $\mathcal{G}_0 \times \mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_{n-1}$, where $\mathcal{G}_i$ is the one-point reflexive frame, for each $3 \leq i < n$. Thus, by $L \subseteq \mathbf{S5}^n$, $\mathcal{F}_{\mathfrak{A},n}$ is a frame for $L$.

Finally, if (ii) holds, that is, if $\neg\varphi_{\mathfrak{A}}$ does not belong to $L$ then, by $\mathbf{K}^n \subseteq L$, $\varphi_{\mathfrak{A}}$ is satisfiable in an $n$-ary product frame. Thus (iii) follows, by Claim 7. □

*Proof of Lemma 6.* This follows from a chain of known results in algebraic logic and duality between Kripke frames and Boolean algebras with operators. Here we only list these results, but in Appendix B below we give the proofs in a modal logic setting. For notions not defined here as well as a detailed summary of properties of relation algebras and connections with cylindric algebras, consult Maddux [13].

As it was introduced in Monk [15], given a finite simple relation algebra $\mathfrak{A}$ as above, one may construct a 3-dimensional *cylindric algebra*

$$\mathsf{Ca}_3\mathfrak{A} = (B, +, \cdot, -, 1, 0, c_i, d_{ij})_{i,j<3}$$

as follows: $(B, +, \cdot, -, 1, 0)$ is the Boolean set algebra of all subsets of $\mathcal{T}_{\mathfrak{A}}$; for each $i < 3$, the unary operation $c_i$ — the $i^{\text{th}}$ *cylindrification* — is defined by

$$c_i X = \{t \in \mathcal{T}_{\mathfrak{A}} \ : \ \exists t' \in X \text{ with } t_i = t'_i\}, \quad \text{for all } X \subseteq \mathcal{T}_{\mathfrak{A}};$$

and the *diagonal elements*, for $i, j < 3$, are

$$d_{ij} = \{t \in \mathcal{T}_{\mathfrak{A}} \ : \ t_k \leq 1'\},$$

where $k < 3$, $k \neq i, j$ and $1'$ is the identity element of the relation algebra $\mathfrak{A}$.

Then the frame $\mathcal{F}_{\mathfrak{A},3}$ above is the atom structure of the diagonal-free reduct $\mathsf{Df}_3\mathfrak{A}$ of $\mathsf{Ca}_3\mathfrak{A}$. Further, $\mathsf{Df}_3\mathfrak{A}$ is clearly finite and $c_0c_1c_2X = 1$ hold in $\mathsf{Df}_3\mathfrak{A}$, for all $X \neq 0$, by property (1) of $\mathcal{F}_{\mathfrak{A},3}$. Thus $\mathsf{Df}_3\mathfrak{A}$ is simple.

It was shown in Monk [15] that

> $\mathfrak{A}$ is representable (as a relation algebra) iff $\mathsf{Ca}_3\mathfrak{A}$ is representable (as a cylindric algebra). Further, $\mathfrak{A}$ is representable with a finite base iff $\mathsf{Ca}_3\mathfrak{A}$ is representable with a finite base.

Since $\mathsf{Df}_3\mathfrak{A}$ is a reduct of a 3-dimensional cylindric algebra and generated by binary elements, the following statement holds (see Johnson [9], Halmos [6], cf. also [7, Thm.5.1.51]):

> $\mathsf{Ca}_3\mathfrak{A}$ is representable (as a cylindric algebra) iff $\mathsf{Df}_3\mathfrak{A}$ is representable (as a diagonal-free cylindric algebra). Further, $\mathsf{Ca}_3\mathfrak{A}$ is representable with a finite base iff $\mathsf{Df}_3\mathfrak{A}$ is representable with a finite base.

Finally, since $\mathsf{Df}_3\mathfrak{A}$ is finite and simple, from basic duality theory we have:

> $\mathsf{Df}_3\mathfrak{A}$ is representable with base $(U, V, W)$ (i.e., embeddable into the diagonal-free cylindric set algebra of all subsets of $U \times V \times W$) iff its atom structure $\mathcal{F}_{\mathfrak{A},3}$ is a p-morphic image of the universal product $\mathbf{S5}^3$-frame on $U \times V \times W$.

Now the lemma clearly follows. $\square$

*Proof of Theorem 4.* Take some finite, simple, representable relation algebra $\mathfrak{A}$ which is representable only with an infinite base (e.g., the *linear* or *point relation algebra*, cf. Maddux [13, §2]), and consider the 3-frame $\mathcal{F}_{\mathfrak{A},3}$ and the 3-modal formula $\varphi_{\mathfrak{A}}$. Then, by Lemmas 5 and 6, $\neg\varphi_{\mathfrak{A}}$ is not in $L$. We show that $\neg\varphi_{\mathfrak{A}}$ is valid in all finite $k$-ary product frames, for any $k \geq 3$. Suppose there is some finite product frame satisfying $\varphi_{\mathfrak{A}}$. Then, by Claim 7, $\mathcal{F}_{\mathfrak{A},3}$ is a p-morphic image of some finite universal product $\mathbf{S5}^3$-frame. This contradicts Lemma 6, since $\mathfrak{A}$ is representable only with an infinite base. Note that in Appendix A we demonstrate how such a formula forces an infinite product frame. $\square$

## Appendix A

Here we give a 6-element 3-frame $\mathcal{F}$ and demonstrate how the Yankov–Fine frame formula of $\mathcal{F}$ can be satisfied in an infinite product frame only. This $\mathcal{F}$ is a simplification of the 3-frame $\mathcal{F}_{\mathfrak{A},3}$ obtained from the linear (point) relation algebra which is used in the proof of Theorem 4. Note that our formula $\Phi$ 'forces' product frames which are infinite rather 'in width' than 'in depth': we will see that if $\Phi$ is true at a point $(x, y, z)$ in a model on a product frame $(U, S_U) \times (V, S_V) \times (W, S_W)$ then the sets $\{u \in U : xS_Uu\}$ and $\{v \in V : yS_Vv\}$ are both infinite.

Let $F$ consist of all permutations of the set $\{0, 1, 2\}$. For $i < 3$, define $R_i$ as "forgetting about $i$ in the triples", that is, for $p, q \in F$, let $pR_iq$ iff

$$(p(j) < p(k) \text{ iff } q(j) < q(k)), \quad \text{whenever } \{i, j, k\} = \{0, 1, 2\},$$

and let $\mathcal{F} = (F, R_0, R_1, R_2)$. Throughout, given some $p \in F$, we write $p_i$ for $p^{-1}(i)$ and identify $p$ with the triple $p_0p_1p_2$, cf. Figure 1. Also, we use notation $p = {*}i{*}j{*}$, whenever $p(i) < p(j)$ holds.

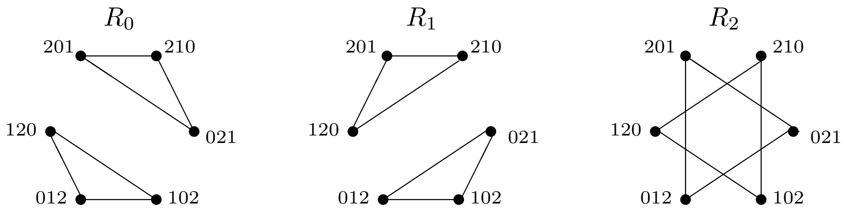



FIGURE 1. The 6-element 3-frame $\mathcal{F}$.

Then the $R_i$ are clearly equivalence relations and it is not hard to see that $\mathcal{F}$ has property (1). Let $\Phi$ be the Yankov–Fine frame formula of $\mathcal{F}$:

$$\Box^+ \bigvee_{p\in F} (p \wedge \neg \bigvee_{p'\neq p} p') \;\wedge\; \Box^+ \bigwedge_{\substack{i<3,p,p'\in F\\ pR_ip'}} (p \to \Diamond_i p') \;\wedge\; \Box^+ \bigwedge_{\substack{i<3,p,p'\in F\\ \neg(pR_ip')}} (p \to \neg\Diamond_i p').$$

CLAIM 9. *There is a product frame satisfying* $\Phi$.

*Proof.* Let $Q_0$, $Q_1$ and $Q_2$ be three pairwise disjoint, dense subsets of the rationals. Take the universal product $\mathbf{S5}^3$-frame $(Q_0, Q_1, Q_2)$ and define a valuation $\upsilon$ of the variables as follows:

$$\upsilon(p) = \{(x_0, x_1, x_2) \in Q_0 \times Q_1 \times Q_2 \;:\; x_{p_0} < x_{p_1} < x_{p_2}\}.$$

Now let $\mathfrak{M} = (Q_0, Q_1, Q_2, \upsilon)$. It is not hard to check that $\mathfrak{M}, (x_0, x_1, x_2) \models \Phi$, for any $(x_0, x_1, x_2)$. $\Box$

CLAIM 10. *Any product frame satisfying* $\Phi$ *is infinite.*

*Proof.* Let $\mathfrak{M}$ be a model on the product frame $(U, S_U) \times (V, S_V) \times (W, S_W)$. We write $xyz$ rather than $(x, y, z)$ for points of $\mathfrak{M}$. Suppose that $x_0 \in U$, $y_0 \in V$, $z_0 \in W$ are such that

$$\mathfrak{M}, x_0y_0z_0 \models \Phi \quad \text{and, say,} \quad \mathfrak{M}, x_0y_0z_0 \models 201. \tag{5}$$

We will show that both $U$ and $V$ must be infinite sets. Let $0 < n < \omega$ and assume inductively that we have defined points $x_i \in U$ and $y_i \in V$ for each $i < n$ satisfying:

$$x_0S_Ux_i \quad \text{and} \quad y_0S_Vy_i, \quad \text{for} \quad 0 < i < n,$$

$$x_i \neq x_j \quad \text{and} \quad y_i \neq y_j, \quad \text{for} \quad i, j < n,\ i \neq j,$$

$$\mathfrak{M}, x_iy_jz_0 \models 201, \quad \text{for} \quad i \leq j < n, \tag{6}$$

$$\mathfrak{M}, x_iy_jz_0 \models 210, \quad \text{for} \quad j < i < n. \tag{7}$$

We will now define $x_n$ and $y_n$. First consider $x_n$. We have $201R_0210$ and, by (6), $\mathfrak{M}, x_0y_{n-1}z_0 \models 201$. By (5), there is some $x_n \in U$ such that

$$x_0S_Ux_n \quad \text{and} \quad \mathfrak{M}, x_ny_{n-1}z_0 \models 210. \tag{8}$$

By (5) and (6), $x_n \neq x_i$, for $i < n$. We show that

$$\mathfrak{M}, x_ny_iz_0 \models 210, \quad \text{for all} \quad i < n - 1, \tag{9}$$

must hold (cf. Figure 2). We need the following claim:

*Claim* 10.1. There are no points $u_0, u_1 \in U$ and $v_0, v_1 \in V$ such that

- $\mathfrak{M}, u_0v_0z_0 \models 210$ and $\mathfrak{M}, u_1v_1z_0 \models 210$,
- $\mathfrak{M}, u_0v_1z_0 \models 201$ and $\mathfrak{M}, u_1v_0z_0 \models 201$,

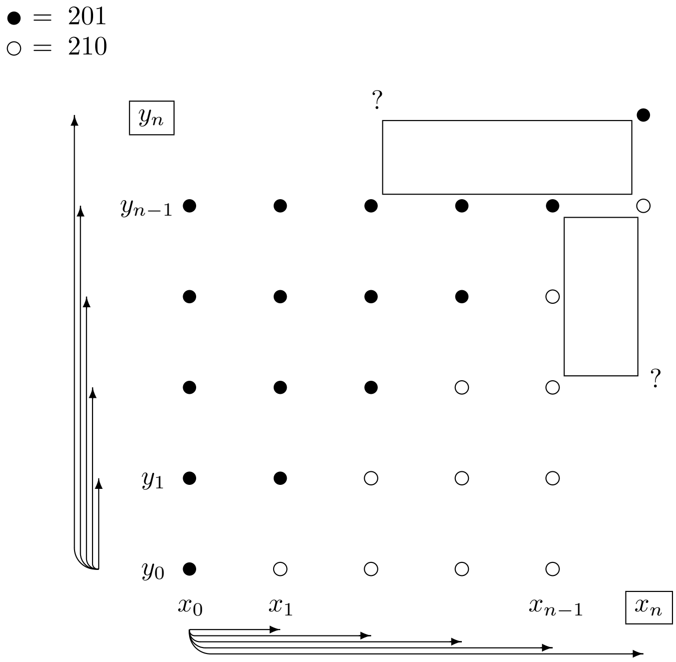



FIGURE 2. The points $x_n$ and $y_n$.

and, for each $i < 2$,

- either $u_i = x_0$ or $x_0 S_U u_i$, and
- either $v_i = y_0$ or $y_0 S_V v_i$.

*Proof of Claim* 10.1. Assume $u_0, u_1, v_0, v_1$ are as above. We will use (5) all the time (cf. Figure 3). Since $\mathfrak{M}, u_0 v_1 z_0 \models 201$ and $201 R_2 021$, thus there is some $z \in W$ such that $z_0 S_W z$ and

(10) $$\mathfrak{M}, u_0 v_1 z \models 021.$$

Then $\mathfrak{M}, u_0 y_0 z \models a$, for some $a \in F$ with $a = {*}0{*}2{*}$, which implies

(11) $$\mathfrak{M}, u_0 v_0 z \models b, \quad \text{for some } b \in F \text{ with } b = {*}0{*}2{*}\,.$$

On the other hand, $\mathfrak{M}, u_0 v_0 z_0 \models 210$ by assumption. Thus $b = {*}1{*}0{*}$, which implies

(12) $$b = 102,$$

by (11). Further, by (10), $\mathfrak{M}, x_0 v_1 z \models c$, for some $c \in F$ with $c = {*}2{*}1{*}$. Thus $\mathfrak{M}, u_1 v_1 z \models d$, for some $d \in F$ with $d = {*}2{*}1{*}$. On the other hand, since by assumption $\mathfrak{M}, u_1 v_1 z_0 \models 210$, $d = {*}1{*}0{*}$ must hold, thus $d = 210$. Therefore, $\mathfrak{M}, u_1 y_0 z \models e$, for some $e \in F$ with $e = {*}2{*}0{*}$. Thus $\mathfrak{M}, u_1 v_0 z \models f$, for some $f \in F$ with $f = {*}2{*}0{*}$. By (12), we have $\mathfrak{M}, x_0 v_0 z \models g$, for some $g \in F$ with $g = {*}1{*}2{*}$. Therefore $f = {*}1{*}2{*}$, and thus $f = 120$ must hold. Finally, this implies that $\mathfrak{M}, u_1 v_0 z_0 \models h$, for some $h \in F$ with $h = {*}1{*}0{*}$, which contradicts the assumption $\mathfrak{M}, u_1 v_0 z_0 \models 201$. □

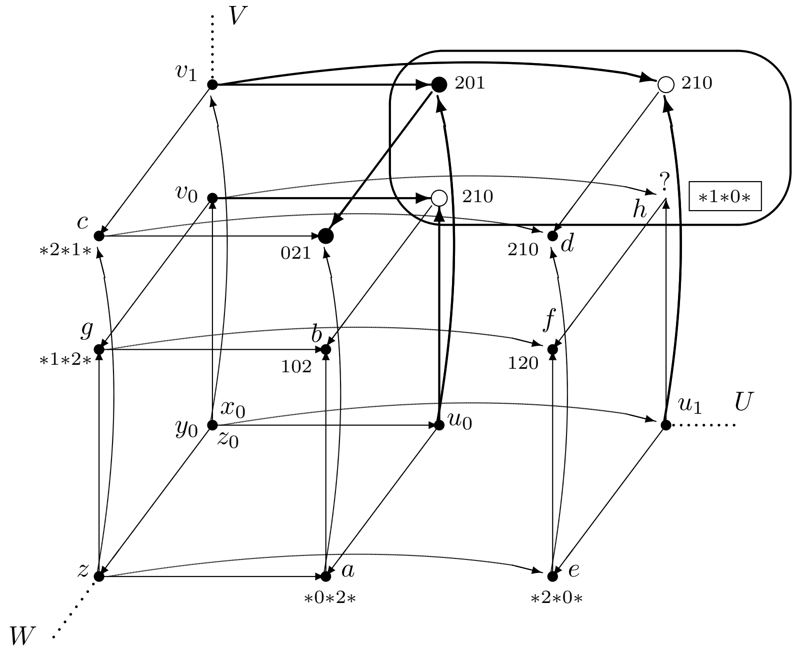



FIGURE 3. The proof of Claim 10.1.

Now one can prove (9) as follows. Take some $i < n-1$. Then we have $\mathfrak{M}, x_{n-1}y_iz_0 \models 210$, by (7). Therefore $\mathfrak{M}, x_0y_iz_0 \models k$, for some $k \in F$ with $k = *2*1*$. Thus $\mathfrak{M}, x_ny_iz_0 \models \ell$, for some $\ell \in F$ with $\ell = *2*1*$. On the other hand, by (8), $\mathfrak{M}, x_ny_0z_0 \models m$, for some $m \in F$ with $m = *2*0*$, thus $\ell = *2*0*$ must hold. Therefore, either $\ell = 201$ or $\ell = 210$. Now apply Claim 10.1 with $u_0 = x_{n-1}$, $u_1 = x_n$, $v_0 = y_i$ and $v_1 = y_{n-1}$ to obtain $\ell = 210$.

Next we define $y_n$. We have $210R_1201$ and we have just shown $\mathfrak{M}, x_ny_0z_0 \models 210$. By (5), there is some $y_n \in V$ such that

$$y_0 S_V y_n \quad \text{and} \quad \mathfrak{M}, x_ny_nz_0 \models 201. \tag{13}$$

By (5), (8) and (9), $y_n \neq y_i$, for $i < n$. It remains to show that, for all $i < n$, $\mathfrak{M}, x_iy_nz_0 \models 201$ hold as well. To this end, take some $i < n$. By (13), we have $\mathfrak{M}, x_0y_nz_0 \models p$, for some $p \in F$ with $p = *2*1*$. Thus $\mathfrak{M}, x_iy_nz_0 \models q$, for some $q \in F$ with $q = *2*1*$. On the other hand, $q = *2*0*$ by (6) and (7), thus either $q = 201$ or $q = 210$. Now apply Claim 10.1 with $u_0 = x_i$, $u_1 = x_n$, $v_0 = y_n$ and $v_1 = y_{n-1}$ to obtain $q = 201$. This way we showed that both $U$ and $V$ must be infinite sets, which completes the proof of Claim 10. □

## APPENDIX B

In order to make the paper self-contained, here we prove Lemma 6 (via Claims 11–14), using notions of modal logic only. However, we would like to emphasize that these proofs are just 'modal mirror images' of the algebraic proofs of Halmos [6], Johnson [9] and Monk [15].

CLAIM 11. *If the (finite, simple) relation algebra $\mathfrak{A}$ is representable with base $U$ then the 3-frame $\mathcal{F}_{\mathfrak{A},3}$ is a p-morphic image of the universal product $\mathbf{S5}^3$-frame on $U \times U \times U$.*

*Proof.* Assume there is some function *rep* embedding $\mathfrak{A}$ into the relation algebra of all subsets of $U \times U$. Define the following function $h$ from $U \times U \times U$ to the set $\mathcal{T}_{\mathfrak{A}}$ of consistent triples of atoms of $\mathfrak{A}$:

$$h(u_0u_1u_2) = t_0t_1t_2 \text{ iff } (u_0, u_1) \in rep(t_2),\ (u_2, u_0) \in rep(t_1),\ (u_1, u_2) \in rep(t_0).$$

It is easy to check that $h$ is well-defined, and a p-morphism onto $\mathcal{F}_{\mathfrak{A},3}$. □

Take a finite simple relation algebra $\mathfrak{A}$ and define, for each $i < j < 3$, a subset $E_{ij}$ of the set $\mathcal{T}_{\mathfrak{A}}$ as follows. Let $k < 3$ be different from both $i$ and $j$. Then take

$$E_{ij} = \{t \in \mathcal{T}_{\mathfrak{A}} \ :\ t_k \leq 1'\}.$$

(Recall that $1'$ denotes the identity element of $\mathfrak{A}$.) It is not hard to see that the following properties hold, for all $i < j < 3$:

(14) $(\forall t \in \mathcal{T}_{\mathfrak{A}})(\exists t', t'' \in E_{ij})\ tR_it' \text{ and } tR_jt''$

(15) $(\forall t, t' \in \mathcal{T}_{\mathfrak{A}})\ t \in E_{ij} \text{ and } tR_kt' \text{ implies } t' \in E_{ij} \quad (k < 3, k \neq i, j)$

(16) $E_{01} \cap E_{02} \subseteq E_{12},\quad E_{01} \cap E_{12} \subseteq E_{02},\quad E_{02} \cap E_{12} \subseteq E_{01}$

(17) $(\forall t, t' \in E_{ij})\ tR_it' \text{ or } tR_jt' \text{ implies } t = t'.$

CLAIM 12. *Assume that there is a p-morphism $h$ from a universal product $\mathbf{S5}^3$-frame $(U_0, U_1, U_2)$ onto $\mathcal{F}_{\mathfrak{A},3}$. Let $U$ be the disjoint union of the sets $U_i$, $i < 3$. Then there is a p-morphism $f$ from the universal product $\mathbf{S5}^3$-frame $(U, U, U)$ onto $\mathcal{F}_{\mathfrak{A},3}$ such that*

(18) $(\forall u_0u_1u_2 \in U \times U \times U)(\forall i < j < 3)\ u_i = u_j \ \textit{ implies } \ f(u_0u_1u_2) \in E_{ij}.$

*Proof.* Note first that for any triple of surjections $f_i : U \to U_i$ $(i < 3)$, the map $f$ defined by

$$f(u_0u_1u_2) = h(f_0(u_0)f_1(u_1)f_2(u_2))$$

is a p-morphism from $(U, U, U)$ onto $\mathcal{F}_{\mathfrak{A},3}$. We will define surjections $f_i : U \to U_i$ $(i < 3)$ such a way that (18) holds.

We claim that for every $u_0 \in U_0$ there is a point $g^{u_0} = u_0u_1u_2$ in $U_0 \times U_1 \times U_2$ such that $h(g^{u_0}) \in E_{01} \cap E_{02} \cap E_{12}$. Indeed, take any $u_0xy \in U_0 \times U_1 \times U_2$. By (14), there is a $u_1 \in U_1$ with $h(u_0u_1y) \in E_{01}$, and there is a $u_2 \in U_2$ with $h(u_0u_1u_2) \in E_{12}$. By (15), $h(u_0u_1u_2) \in E_{01}$ also holds, and so, by (16), $h(u_0u_1u_2) \in E_{02}$. In the same way we can show that for every $u_1 \in U_1$ ($u_2 \in U_2$) there is $g^{u_1} = u_0u_1u_2$ (respectively, $g^{u_2} = u_0u_1u_2$) in $U_0 \times U_1 \times U_2$ such that $h(g^{u_1}) \in E_{01} \cap E_{02} \cap E_{12}$ (and $h(g^{u_2}) \in E_{01} \cap E_{02} \cap E_{12}$).

Define maps $f_i$ from $U$ onto $U_i$ $(i < 3)$ by taking $f_i(u)$ to be the $i$-th coordinate of $g^u$, for every $u \in U$. (Since $f_i$ is the identity on $U_i$, $f_i$ is surjective.) Define $f : U \times U \times U \to \mathcal{T}_{\mathfrak{A}}$ as above. We show that $f$ satisfies (18). For any $u \in U$,

$$f(uuu) = h(f_0(u)f_1(u)f_2(u)) = h(g^u) \in E_{01} \cap E_{02} \cap E_{12}.$$

For any $v \in U$, $f(uuv) \in E_{01}$, $f(uvu) \in E_{02}$ and $f(vuu) \in E_{12}$ follow, by (15). □

Claim 13. *Assume there is a p-morphism $f$ from some universal product $\mathbf{S5}^3$-frame $(U,U,U)$ onto $\mathcal{F}_{\mathfrak{A},3}$ such that* (18) *holds. Then there is some set $V$ with $|V| \leq |U|$ and a p-morphism $g$ from $(V,V,V)$ onto $\mathcal{F}_{\mathfrak{A},3}$ such that*

$$(\forall v_0v_1v_2 \in V \times V \times V)(\forall i < j < 3)\ v_i = v_j \quad \textit{iff} \quad g(v_0v_1v_2) \in E_{ij}. \tag{19}$$

*Proof.* For every $i < j < 3$, we define a relation $\mathcal{D}_{ij} \subseteq U \times U$ by taking

$$\mathcal{D}_{ij} = \{(x,y) \in U \times U : \exists u_0,u_1,u_2 \in U \text{ with } u_i = x,\ u_j = y \text{ and } f(u_0u_1u_2) \in E_{ij}\}.$$

In fact, these three relations coincide. Let us check, for instance, that we have $\mathcal{D}_{01} \subseteq \mathcal{D}_{02}$. Suppose $f(xyz) \in E_{01}$. By (15), $f(xyy) \in E_{01}$ and by (18), $f(xyy) \in E_{12}$. It follows then from (16) that $f(xyy) \in E_{02}$.

Let $\mathcal{D}$ denote the relation $\mathcal{D}_{01} = \mathcal{D}_{02} = \mathcal{D}_{12}$. We show that $\mathcal{D}$ is an equivalence relation on $U$. By (18) it is reflexive. To show symmetry, let $f(xyz) \in E_{01}$. By (15), $f(xyx) \in E_{01}$ as well. On the other hand, $f(xyx) \in E_{02}$, by (18). Thus, by (16), $f(xyx) \in E_{12}$ which implies $(y,x) \in \mathcal{D}$. To prove transitivity, suppose $x\mathcal{D}_{01}y$ and $y\mathcal{D}_{12}z$. Thus $f(xys) \in E_{01}$ and $f(ryz) \in E_{12}$, for some $s$ and $r$. Therefore, by (15), $f(xyz) \in E_{01} \cap E_{12}$ and, by (16), $f(xyz) \in E_{02}$, that is, $x\mathcal{D}_{02}z$.

Denote by $[u]$ the $\mathcal{D}$-equivalence class containing $u$. Let $V = \{[u] : u \in U\}$. Define the function $g : V \times V \times V \to \mathcal{T}_{\mathfrak{A}}$ by taking

$$g([u_0][u_1][u_2]) = f(u_0u_1u_2).$$

We show that this $g$ is well-defined: If $u_i\mathcal{D}v_i$, for each $i < 3$, then $f(u_0u_1u_2) = f(v_0v_1v_2)$ holds. We prove first that $f(u_0u_1u_2) = f(u_0v_1u_2)$, if $u_1\mathcal{D}v_1$. We do this by showing that, for each $i < 3$, $f(u_0u_1u_2)_i = f(u_0v_1u_2)_i$, i.e., they are the same atom of $\mathfrak{A}$. For $i = 1$ it is obvious by the definition of $\mathcal{F}_{\mathfrak{A},3}$. Next, let $i = 2$. By (14), there is some $x \in U$ with $f(u_0u_1x) \in E_{12}$, thus $u_1\mathcal{D}x$ follows, which implies $v_1\mathcal{D}x$. Thus there is some $y \in U$ with $f(yv_1x) \in E_{12}$. By (15), $f(u_0v_1x) \in E_{12}$ also holds, thus, by (17), $f(u_0u_1x) = f(u_0v_1x)$. Therefore,

$$f(u_0u_1u_2)_2 = f(u_0u_1x)_2 = f(u_0v_1x)_2 = f(u_0v_1u_2)_2.$$

The case of $i = 0$ is analogous. Further, it can be shown similarly that $f(u_0v_1u_2) = f(v_0v_1u_2)$ and $f(v_0v_1u_2) = f(v_0v_1v_2)$ also hold, which proves that $g$ is well-defined.

It is obvious by its definition that $g$ is a p-morphism onto $\mathcal{F}_{\mathfrak{A},3}$ satisfying (19). □

Claim 14. *Assume there is a p-morphism $g$ from some universal product $\mathbf{S5}^3$-frame $(V,V,V)$ onto $\mathcal{F}_{\mathfrak{A},3}$ such that* (19) *holds. Then the relation algebra $\mathfrak{A}$ is representable with base $V$, that is, $\mathfrak{A}$ is embeddable into the set relation algebra of all subsets of $V \times V$.*

*Proof.* Recall that the points of $\mathcal{F}_{\mathfrak{A},3}$ are the consistent triples of atoms of $\mathfrak{A}$. Define the representation *rep* of $\mathfrak{A}$ with base $V$ as follows: For each atom $c$ of $\mathfrak{A}$, take

$$rep(c) = \{(u,v) \in V \times V\ :\ \exists w \in V \text{ with } g(uvw)_2 = c\}.$$

Then, by the definition of $\mathcal{F}_{\mathfrak{A},3}$, $rep(c_1)$ and $rep(c_2)$ are disjoint, whenever $c_1 \neq c_2$. Extend *rep* to any element $x$ of $\mathfrak{A}$ by taking

$$rep(x) = \bigcup\{rep(c)\ :\ c \text{ is an atom of } \mathfrak{A} \text{ and } c \leq x\}.$$

It is straightforward to check that *rep* is a Boolean embedding. We show that it is a relation algebra homomorphism. First, $rep(1') = \{(u,u) : u \in V\}$ holds because of (19). Since ; and $\breve{}$ distribute over Boolean join, it is enough to show that *rep* preserves ; and $\breve{}$ for atoms. To this end, we need the following claim:

*Claim* 14.1. *For all $u, v, w \in V$ and atoms $a, b, c$ of $\mathfrak{A}$,*

$$g(uvw) = abc \quad \textit{iff} \quad (u, v) \in rep(c),\ (v, w) \in rep(a) \text{ and } (w, u) \in rep(b).$$

*Proof of Claim 14.1.* We use the following property of $\mathcal{F}_{\mathfrak{A},3}$ all the time. For all $t \in \mathcal{T}_{\mathfrak{A}}$, $i < j < 3$ and $k < 3$ with $k \neq i, j$,

$$t \in E_{ij} \implies t_k \leq 1' \implies t_i = t_j^{\smile}. \tag{20}$$

Suppose that $g(uvw) = abc$. Then $(u, v) \in rep(c)$ by definition. In order to prove $(v, w) \in rep(a)$, we show — with the help of (19) and (20) — that $g(vwu) = bca$:

$$\begin{array}{lll}
g(uvw) = abc & g(uvw) = abc & g(uvw) = abc \\
\quad R_1 & \quad R_2 & \quad R_0 \\
g(uww) = *bb^{\smile} & g(uvu) = c^{\smile}*c & g(vvw) = aa^{\smile}* \\
\quad R_2 & \quad R_0 & \quad R_1 \\
g(uwu) = b*b^{\smile} & g(vvu) = c^{\smile}c* & g(vww) = *a^{\smile}a \\
\quad R_0 & \quad R_1 & \quad R_2 \\
g(vwu) = b** & g(vwu) = *c* & g(vwu) = **a.
\end{array}$$

Similarly, one can show $g(wuv) = cab$, thus $(w, u) \in rep(b)$. For the other direction, by (20) we know that $g(wuw) = b^{\smile}*b$ and $g(vww) = *a^{\smile}a$, thus again an argument similar to the above proves $g(uvw) = abc$. □

Using (20) and Claim 14.1, it is not hard to check that $rep(c)^{\smile} = rep(c^{\smile})$ and $rep(c_1\,;c_2) = rep(c_1)\,;rep(c_2)$ hold, for any atoms $c, c_1, c_2$. □

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