A note on axiomatisations of two-dimensional modal logics

Agi Kurucz

Department of Informatics
King's College London
agi.kurucz@kcl.ac.uk

Abstract. We analyse the role of the modal axiom corresponding to the first-order formula "$\exists y\,(x = y)$" in axiomatisations of two-dimensional propositional modal logics.

One of the several possible connections between propositional multi-modal logics and classical first-order logic is to consider finite variable fragments of the latter as 'multi-dimensional' modal formalisms: First-order variable-assignment tuples are regarded as possible worlds in Kripke frames, and each first-order quantification $\exists v_i$ and $\forall v_i$ as 'coordinate-wise' modal operators $\Diamond_i$ and $\Box_i$ in these frames. This view is implicit in the algebraisation of finite variable fragments using finite dimensional cylindric algebras [6], and is made explicit in [15, 12].

Here we look at axiomatisation questions for the two-dimensional case from this modal perspective. (For basic notions in modal logic and its Kripke semantics, consult e.g. [2, 3].) We consider the propositional multi-modal language $\mathcal{ML}_2^\delta$ having the usual Boolean operators, unary modalities $\Diamond_0$ and $\Diamond_1$ (and their duals $\Box_0$, $\Box_1$), and a constant δ:

$$\mathcal{ML}_2^\delta : \qquad p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond_0\varphi \mid \Diamond_1\varphi \mid \delta$$

Formulas of this language can be embedded into the two-variable fragment of first-order logic by mapping propositional variables to binary atoms $P(v_0, v_1)$ (with this fixed order of the two available variables), diamonds $\Diamond_i$ to quantification $\exists v_i$, and the 'diagonal' constant δ to the equality atom $v_0 = v_1$. Semantically, we look at first-order models as multimodal Kripke frames (fitting to the above language) of the form

$$\begin{array}{lll} \langle U \times U, \equiv_0, \equiv_1, Id\rangle, & & \text{where, for all } u_0, u_1, v_0, v_1 \in U, \\ \langle u_0, u_1\rangle \equiv_0 \langle v_0, v_1\rangle & \text{iff} & u_1 = v_1, \\ \langle u_0, u_1\rangle \equiv_1 \langle v_0, v_1\rangle & \text{iff} & u_0 = v_0, \text{ and} \\ Id = \{\langle u, u\rangle : u \in U\}. & & \end{array}$$

We call frames of this kind *square frames*. The above embedding is validity-preserving in the sense that a modal $\mathcal{ML}_2^\delta$-formula φ is valid in all square frames iff its translation $\varphi^\dagger$ is a first-order validity.

In the algebraic setting, the modal logic of square-frames corresponds to the equational theory of the variety RCA_2 of *2-dimensional representable cylindric algebras*. The equational theory of RCA_2 is well-known to be finitely axiomatisable [6]. By turning this equational axiomatisation to modal $\mathcal{ML}_2^\delta$-formulas, we obtain a finite axiomatisation of the modal logic of square frames [15]. In order to 'deconstruct' this axiomatisation and to try to analyse which axiom is responsible for which properties of the modal logic of square frames, below we list these axioms divided into two groups:

(i) Unimodal properties describing individual modal operators, for $i = 0, 1$:

$$\Box_i p \to p \qquad \Box_i p \to \Box_i \Box_i p \qquad \Diamond_i p \to \Box_i \Diamond_i p \tag{1}$$

These are the (Sahlqvist) axioms of the well-known modal logic **S5**, saying that each $\equiv_i$ is an equivalence relation.

(ii) Multimodal, 'dimension-connecting' properties, describing the interactions between the two diamonds, and between the diamonds and the diagonal constant:

$$\Diamond_0 \Diamond_1 p \leftrightarrow \Diamond_1 \Diamond_0 p \tag{2}$$

$$\Diamond_0 \delta \wedge \Diamond_1 \delta \tag{3}$$

$$\big(\Diamond_0(\delta \wedge p) \to \Box_0(\delta \to p)\big) \wedge \big(\Diamond_1(\delta \wedge p) \to \Box_1(\delta \to p)\big) \tag{4}$$

$$\delta \wedge \Diamond_0(\neg p \wedge \Diamond_1 p) \to \Diamond_1(\neg\delta \wedge \Diamond_0 p) \tag{5}$$

$$\delta \wedge \Diamond_1(\neg p \wedge \Diamond_0 p) \to \Diamond_0(\neg\delta \wedge \Diamond_1 p) \tag{6}$$

These axioms are also Sahlqvist formulas, with easily computable first-order correspondents: Axiom (2) says that $\equiv_0$ and $\equiv_1$ commute, (3) says that at each 'horizontal' and 'vertical' coordinate, there is at least one 'diagonal' point, while (4) says that there is at most one such. Finally, (5) is a kind of generalisation of (2) when we start from a 'diagonal' point: It says that if we start with a $\equiv_0$-step, then move on to a different point by a $\equiv_1$-step, then we can always complete the same journey by taking first a $\equiv_1$-step to a 'non-diagonal' point, followed by a $\equiv_0$-step. (And (6) says the same about starting with a $\equiv_1$-step, and then taking a $\equiv_0$ one.) Note that the axiomatisation given in [6] contains slightly complicated forms of (5) and (6). As it is shown by Venema [15], on the basis of (1), (2) and (4), the 'Henkin-axioms' are equivalent to (5) and (6).

One of the motivations in the study of so-called two-dimensional modal logics is to understand how much influence each of the (i)- and (ii)-like properties has on the resulting logics. Below we consider Kripke structures where

- the set of possible worlds is still a full Cartesian product of two sets, and the relations between the pairs of points still 'act coordinate-wise' (so at least (2), but possibly further properties in (ii) still hold),
- the accessibility relations between the pairs of points are not necessarily equivalence relations (so (i) might not hold).

Note that this direction is kind of orthogonal to the one taken by relativised cylindric algebras [6, 7] and guarded fragments of first-order logic [1], where (i) is kept unchanged, while generalisations of (ii) are considered.

Let us introduce a 'product-like' construction on Kripke frames. This and similar constructions were first considered by Segerberg [13] and Shehtman [14], see also [4, 9, 8]. Given unimodal Kripke frames $\mathfrak{F}_0 = \langle U_0, R_0\rangle$ and $\mathfrak{F}_1 = \langle U_1, R_1\rangle$, their *δ-product* is the multimodal frame

$$\mathfrak{F}_0 \times^\delta \mathfrak{F}_1 = \langle U_0 \times U_1, \bar{R}_0, \bar{R}_1, Id\rangle,$$

where $U_0 \times U_1$ is the Cartesian product of sets U_0 and U_1, the binary relations $\bar{R}_0$ and $\bar{R}_1$ are defined by taking,

$$\begin{array}{lll} \langle u_0, u_1\rangle\, \bar{R}_0 \langle v_0, v_1\rangle & \text{iff} & u_1 = v_1 \text{ and } u_0 R_0 v_0, \\ \langle u_0, u_1\rangle\, \bar{R}_1 \langle v_0, v_1\rangle & \text{iff} & u_0 = v_0 \text{ and } u_1 R_1 v_1, \end{array}$$

and

$$Id = \{\langle u, u\rangle : u \in U_0 \cap U_1\}.$$

Observe that if $\mathfrak{F} = \langle U, U \times U\rangle$ is an universal frame, then $\mathfrak{F} \times^\delta \mathfrak{F}$ is a square frame. Let us introduce some notation for logics of some special classes of δ-product frames:

$$\begin{aligned} \mathbf{K} \times^\delta \mathbf{K} &= \{\varphi \in \mathcal{ML}_2^\delta : \varphi \text{ is valid in } \mathfrak{F}_0 \times^\delta \mathfrak{F}_1,\ \mathfrak{F}_i \text{ are arbitrary frames}\}, \\ \mathbf{K} \times^{sq} \mathbf{K} &= \{\varphi \in \mathcal{ML}_2^\delta : \varphi \text{ is valid in } \mathfrak{F} \times^\delta \mathfrak{F},\ \mathfrak{F} \text{ is an arbitrary frame}\}, \\ \mathbf{S5} \times^\delta \mathbf{S5} &= \{\varphi \in \mathcal{ML}_2^\delta : \varphi \text{ is valid in } \mathfrak{F}_0 \times^\delta \mathfrak{F}_1,\ \mathfrak{F}_i \text{ are equivalence frames}\}, \\ \mathbf{S5} \times^{sq} \mathbf{S5} &= \{\varphi \in \mathcal{ML}_2^\delta : \varphi \text{ is valid in } \mathfrak{F} \times^\delta \mathfrak{F},\ \mathfrak{F} \text{ is an equivalence frame}\} \\ &= \{\varphi \in \mathcal{ML}_2^\delta : \varphi \text{ is valid in } \mathfrak{F} \times^\delta \mathfrak{F},\ \mathfrak{F} \text{ is a universal frame}\} \\ &= \{\varphi \in \mathcal{ML}_2^\delta : \varphi \text{ is valid in all square frames}\}. \end{aligned}$$

Using this notation, the finite axiomatisability of RCA_2 can be reformulated as the following:

Theorem 1. [6] $\mathbf{S5} \times^{sq} \mathbf{S5}$ *is finitely axiomatised by the axioms* (1)–(6).

In this note we investigate the particular role of axiom (3) in this axiomatisation. To begin with, this axiom is quite strong in the sense that it can 'force' the **S5**-properties (1) in the presence of 'two-dimensionality', as the following surprising statement shows:

Theorem 2. [10, 11] *Let L be any canonical modal logic with*

$$\mathbf{K} \times^\delta \mathbf{K} \subseteq L \subseteq \mathbf{S5} \times^{sq} \mathbf{S5}.$$

Then $\mathbf{S5} \times^{sq} \mathbf{S5}$ is finitely axiomatisable over L: $\mathbf{S5} \times^{sq} \mathbf{S5}$ is the smallest modal logic containing L and axiom (3).

In particular, as a consequence we obtain that $\mathbf{S5}\times^{sq}\mathbf{S5}$ =‘$\mathbf{S5}\times^{\delta}\mathbf{S5}$ + (3)’. Here we show that the remaining axioms indeed do axiomatise $\mathbf{S5}\times^{\delta}\mathbf{S5}$:

Theorem 3. $\mathbf{S5}\times^{\delta}\mathbf{S5}$ *is finitely axiomatised by the axioms* (1), (2), (4)–(6).

On the one hand, these axioms are clearly valid in δ-products of equivalence frames. On the other hand, since (1), (2), and (4)–(6) are all Sahlqvist-formulas, the modal logic they axiomatise is determined by a first-order definable class of frames, and so it has the countable frame property. Therefore, it is enough to show the following:

Lemma 4. *Let* $\mathfrak{G} = \langle W, R_0, R_1, D\rangle$ *be a countable rooted frame, validating* (1), (2), *and* (4)–(6). *Then* $\mathfrak{G}$ *is a p-morphic image of a* δ*-product* $\mathfrak{F}_0\times^{\delta}\mathfrak{F}_1$ *for some universal frames* $\mathfrak{F}_i = \langle U_i, U_i \times U_i\rangle$, $i = 0, 1$.

Proof. It is a step-by-step argument that is a generalisation of Venema's [15] proof showing that countable rooted frames validating (1)–(6) are p-morphic images of square frames.

One way of presenting such an argument is by defining a ‘p-morphism game’ $\mathcal{G}_\omega(\mathfrak{G})$ between two players $\forall$ (male) and $\exists$ (female) over $\mathfrak{G}$. In this game, $\exists$ constructs step-by-step, (special) homomorphisms from larger and larger δ-products of universal frames to $\mathfrak{G}$, and $\forall$ tries to challenge her by pointing out possible ‘defects’: reasons why her current homomorphism is not an onto p-morphism yet.

To this end, we call a triple $N = \langle U_0^N, U_1^N, f^N\rangle$ a $\mathfrak{G}$-*network*, if U_0^N, U_1^N are nonempty sets, and $f^N : U_0^N \times U_1^N \to W$ is a function such that the following hold, for all $u_i, v_i \in U_i^N$, $i = 0, 1$:

(nw1) $f^N(u_0, u_1)R_0 f^N(v_0, u_1)$ and $f^N(u_0, u_1)R_1 f^N(u_0, v_1)$,
(nw2) $f^N(u_0, u_1) \in D$ iff $u_0 = u_1 \in U_0^N \cap U_1^N$, and
(nw3) if there exists w in D with $f^N(u_0, u_1)R_i w$, then $u_{1-i} \in U_i^N$.

The two players build a countable sequence of $\mathfrak{G}$-networks

$$N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k \subseteq \ldots .$$

(Here $N_k \subseteq N_{k+1}$ means that $U_i^{N_k} \subseteq U_i^{N_{k+1}}$, $i = 0, 1$, and $f^{N_k} \subseteq f^{N_{k+1}}$.) In round 0, $\forall$ picks any point r in D if there is such. If not, then just any point in W. (As R_0 and R_1 are equivalence relations and $R_0 \cup R_1$ is rooted, any point in W is a root in $\mathfrak{G}$.) $\exists$ responds with the $\mathfrak{G}$-network $U_0^{N_0} = \{u_0\}$, $U_1^{N_0} = \{u_1\}$ and $f^{N_0}(u_0, u_1) = r$, with $u_0 = u_1$ if $r \in D$, and $u_0 \neq u_1$ otherwise.

In round k ($0 < k < \omega$), some sequence $N_0 \subseteq \cdots \subseteq N_{k-1}$ of $\mathfrak{G}$-networks has already been built. $\forall$ picks

- a pair $\langle x, y\rangle \in U_0^{N_{k-1}} \times U_1^{N_{k-1}}$,
- a point $w \in W$, and
- an index $i = 0$ or $i = 1$

such that $f^{N_{k-1}}(x,y)R_iw$ holds. Let us consider $\exists$'s possible responses in the $i=0$ case. (The $i=1$ case is symmetrical.) She can respond in two ways. If there is some $u\in U_0^{N_{k-1}}$ with $f^{N_{k-1}}(u,y)=w$, then she responds with $N_k=N_{k-1}$. Otherwise, she has to respond (if she can) with some $\mathfrak{G}$-network $N_k\supseteq N_{k-1}$ such that $u^*\in U_0^{N_k}$ and $f^{N_k}(u^*,y)=w$, for some fresh point u^*.

We say that $\exists$ *has a winning strategy in* $\mathcal{G}_\omega(\mathfrak{G})$ if she can respond in each round k for $k<\omega$, no matter what moves $\forall$ take in the rounds. It is not hard to see that if $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{G})$, then $\mathfrak{G}$ is a p-morphic image of a a δ-product of universal frames: Consider a play of the game when $\forall$ eventually picks all possible pairs and corresponding R_i-connected points in $\mathfrak{G}$ (since $\mathfrak{G}$ is countable and rooted, he can do this). If $\exists$ uses her strategy, then she succeeds to construct a countable ascending chain of $\mathfrak{G}$-networks whose union gives the required p-morphism.

We show that if $\mathfrak{G}$ validates axioms (1), (2), and (4)–(6), then $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{G})$. The case of round 0 is straightforward. Suppose that we are in round $k>0$ and $\forall$ picks $\langle x,y\rangle$, w, and $i=0$ as above. We omit the case where $\exists$'s response is fully determined by the rules of the game, so we may assume that

$$f^{N_{k-1}}(u,y)\neq w,\ \text{ for all } u\in U_0^{N_{k-1}}. \tag{7}$$

We claim that

$$w\notin D \tag{8}$$

follows. Indeed, if $w\in D$ then $y\in U_0^{N_{k-1}}\cap U_1^{N_{k-1}}$ by (nw3), and therefore $f^{N_{k-1}}(y,y)\in D$ by (nw2). So, by axioms (1) and (4), $w=f^{N_{k-1}}(y,y)$ follows, contradicting (7).

We let $U_0^{N_k}=U_0^{N_{k-1}}\cup\{u^*\}$, for some fresh point u^*, $f^{N_k}(u^*,y)=w$, and $f^{N_k}\supseteq f^{N_{k-1}}$. We consider two cases: either there is no $w^*\in D$ with wR_1w^*, or there is such a w^*.

Case 1. There is no $w^*\in D$ with wR_1w^*.
Then we let $U_1^{N_k}=U_1^{N_{k-1}}$. Take some $u\in U_1^{N_{k-1}}$, $u\neq y$. We need to define $f^{N_k}(u^*,u)$ such that (nw1)–(nw3) hold. We have $f^{N_k}(x,u)R_1f^{N_k}(x,y)R_0w$ by (nw1). So by axiom (2), there exists w_u such that $f^{N_k}(x,u)R_0w_uR_1w$. As w_uR_1w, by axiom (1) there is no $v\in D$ with w_uR_1v, in particular, $w_u\notin D$. Therefore, $f^{N_k}(u^*,u)=w_u$ is a good choice.

Case 2. There exists $w^*\in D$ with wR_1w^*.
Then we let $U_1^{N_k}=U_1^{N_{k-1}}\cup\{u^*\}$. We need to define f^{N_k} on the new pairs such that (nw1)–(nw3) hold. There are several cases (see Fig. 1):

– First, let $f^{N_k}(u^*,u^*)=w^*$.
– Next, take some $u\in U_1^{N_{k-1}}$, $u\neq y$.
  • *Case* (a). There is no $v\in D$ with $f^{N_k}(x,u)R_0v$.
  As by (nw1) we have $f^{N_k}(x,u)R_1f^{N_k}(x,y)R_0w$, by axiom (2) there exists w_u such that $f^{N_k}(x,u)R_0w_uR_1w$. As $f^{N_k}(x,u)R_0w_u$, by axiom (1) there is no $v\in D$ with w_uR_0v, and therefore $f^{N_k}(u^*,u)=w_u\notin D$ will do.

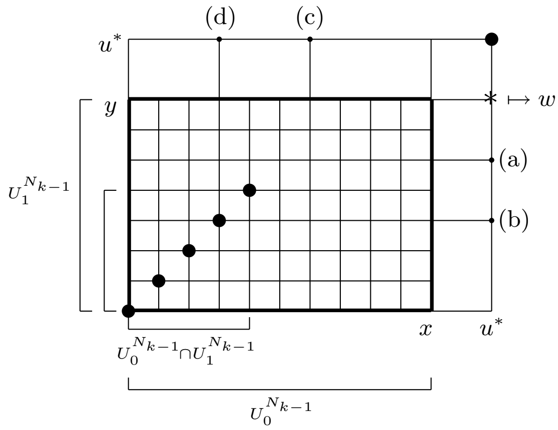

Fig. 1. The subcases in Case 2.

- *Case* (b). There is $v \in D$ with $f^{N_k}(x,u)R_0v$.
  By (nw3) and (nw2), $u \in U_0^{N_{k-1}}$ and $f^{N_k}(u,u) \in D$. By (7), we have $f^{N_k}(u,y) \neq w$. As by (nw1) we also have $f^{N_k}(u,u)R_1f^{N_k}(u,y)R_0w$, by axiom (6) there is $w_u \notin D$ with $f^{N_k}(u,u)R_0w_uR_1w$, and so $f^{N_k}(u^*,u) = w_u \notin D$ will do.

– Finally, take some $u \in U_0^{N_{k-1}}$.

- *Case* (c). There is no $v \in D$ with $f^{N_k}(u,y)R_1v$.
  As by (nw1) we have $f^{N_k}(u,y)R_0wR_1f^{N_k}(u^*,u^*)$, by axiom (2) there is w_u such that $f^{N_k}(u,y)R_1w_uR_0f^{N_k}(u^*,u^*)$. As $f^{N_k}(u,y)R_1w_u$, by axiom (1) there is no $v \in D$ with w_uR_1v, and so $f^{N_k}(u,u^*) = w_u \notin D$ will do.
- *Case* (d). There is $v \in D$ with $f^{N_k}(u,y)R_1v$.
  By (nw3) and (nw2), $u \in U_1^{N_{k-1}}$ and $f^{N_k}(u,u) \in D$. On the one hand, we have $f^{N_k}(u^*,u) \neq f^{N_k}(u^*,u^*)$, as $f^{N_k}(u^*,u) \notin D$ by Case (b) and (8), while $f^{N_k}(u^*,u^*) \in D$ by definition. On the other hand, by (nw1) we have $f^{N_k}(u,u)R_0f^{N_k}(u^*,u)R_1f^{N_k}(u^*,u^*)$. So by axiom (5), there is $w_u \notin D$ with $f^{N_k}(u,u)R_1w_uR_0f^{N_k}(u^*,u^*)$. Thus $f^{N_k}(u,u^*) = w_u \notin D$ will do,

completing the proof of Lemma 4.

The role of (3)-like axioms in two-dimensional logics without the individual **S5**-properties is far from clear. Unlike axioms (2) and (4)–(6), axiom (3) does not hold in $\mathbf{K}\times^{sq}\mathbf{K}$. In fact, it is not known whether, say, $\mathbf{K}\times^{sq}\mathbf{K}$ is finitely axiomatisable over $\mathbf{K}\times^{\delta}\mathbf{K}$. Also, though a general argument of [5] can be used to show that both logics are recursively enumerable, no explicit axiomatisations for them are known. Such axiomatisations should be infinite however: As it is shown by Kikot [8], neither $\mathbf{K}\times^{sq}\mathbf{K}$ nor $\mathbf{K}\times^{\delta}\mathbf{K}$ can be axiomatised using finitely many propositional variables.

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