Representable Cylindric Algebras and Many-Dimensional Modal Logics

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The equationally expressible properties of the cylindrifications and the diagonals in finite-dimensional representable cylindric algebras can be divided into two groups:

(i) 'One-dimensional' properties describing individual cylindrifications. These can be fully characterised by finitely many equations saying that each c_i, for $i < n$, is a normal ($c_i 0 = 0$), additive ($c_i(x+y) = c_i x + c_i y$) and complemented closure operator:

$$x \leq c_i x \qquad c_i c_i x \leq c_i x \qquad c_i(-c_i x) \leq -c_i x. \tag{1}$$

(ii) 'Dimension-connecting' properties, that is, equations describing the diagonals and interaction between different cylindrifications and/or diagonals. These properties are much harder to describe completely, and there are many results in the literature on their complexity.

The main aim of this chapter is to study generalisations of (i) while keeping (ii) as unchanged as possible. In other words, we would like to analyse how much of the complexity of RCA_n is due to its 'many-dimensional' character and how much of it to the particular properties of the cylindrifications. Note that this direction is kind of orthogonal to the one taken by *relativised cylindric algebras* [15, Section 5.5], where (i) is kept unchanged, while generalisations of (ii) are considered.

In our investigations we view representable cylindric algebras from the perspective of *multimodal logic*. This approach is explained in detail in Venema's chapter of this volume and in [35]. In particular, we look at n-dimensional cylindric set algebras as subalgebras of the complex algebra of 'n-dimensional' *relational structures* of the form

$$\langle {}^nU, \equiv_i, Id_{ij}\rangle_{i,j<n}, \tag{2}$$

where, for all $i, j < n$, $\mathbf{u}, \mathbf{v} \in {}^nU$,

$$\begin{aligned} \mathbf{u} \equiv_i \mathbf{v} \quad &\iff \quad u_k = v_k \text{ for all } k < n,\ k \neq i, \text{ and} \\ \mathbf{u} \in Id_{ij} \quad &\iff \quad u_i = u_j. \end{aligned}$$

Instead of equations in the algebraic language having operators d_{ij} and c_i, we use formulas of the corresponding propositional multimodal language having modal constants δ_{ij} and unary diamonds $\Diamond_i$ (and their duals $\Box_i$), for $i, j < n$. As the variety RCA_n of n-dimensional representable cylindric algebras is generated by cylindric set algebras, equations valid in RCA_n correspond to multimodal formulas valid in all structures described in (2). The above classification of equational properties now translates to the following classification of modally expressible properties:

(i) Modal formulas saying that each $\Diamond_i$ is normal and distributes over $\vee$, and axioms of modal logic **S5**, for each $i < n$:

$$\Box_i p \to p \qquad \Box_i p \to \Box_i \Box_i p \qquad \Diamond_i p \to \Box_i \Diamond_i p. \tag{3}$$

(ii) Multimodal formulas describing 'dimension-connecting' properties of the n-dimensional structures described in (2).

Our investigations can also be motivated from a purely modal logic point of view. The n-dimensional relational structures described in (2) are examples of *products of Kripke frames*, a notion introduced in [38, 39]. Product frames have been widely used for modelling interactions between modal operators representing time, space, knowledge, actions, etc.; see [8, 31] and references therein. One can also consider this chapter as a demonstration of how cylindric algebraic results and techniques can be used for studying combinations of modal logics.

1 Special varieties of complex algebras

This chapter is not self-contained in the sense that we use without explicit reference standard notions and results from basic modal logic; such as *p-morphisms* (also known as *bounded morphisms* or *zigzag morphisms*), *inner substructures* (also known as *generated subframes*), Sahlqvist formulas and canonicity, and duality between relational structures and *Boolean algebras with operators* (*BAOs*). For notions and statements not defined or proved here, see other chapters of this volume (like that of Hirsch and Hodkinson, and Venema) or [5, 4, 11].

We begin with introducing some notation and terminology. If x is a point in a relational structure $\mathfrak{F}$ then we denote by $\mathfrak{F}^x$ the smallest inner substructure of $\mathfrak{F}$

containing x. We call $\mathfrak{F}^x$ a *point-generated inner substructure* of $\mathfrak{F}$. If $\mathfrak{F} = \mathfrak{F}^x$ for some x, then $\mathfrak{F}$ is called *rooted.* Rooted structures are important in modal logic, as in any model over $\mathfrak{F}$, the truth-values of modal formulas at x depend only on how the model behaves at points in $\mathfrak{F}^x$.

Apart from the usual operators $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ on classes of algebras (see [16, Ch. 0]) we use the following operators on classes of relational structures of the same signature:

$$\begin{aligned}
\mathbb{I}\,\mathcal{C} &= \text{isomorphic copies of inner substructures of structures in } \mathcal{C},\\
\mathbb{I}_{\mathsf{p}}\,\mathcal{C} &= \text{isomorphic copies of point-generated inner substructures}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{of structures in } \mathcal{C},\\
\mathbb{U}\mathsf{p}\,\mathcal{C} &= \text{isomorphic copies of ultraproducts of structures in } \mathcal{C}.
\end{aligned}$$

The (*full*) *complex algebra* of a relational structure $\mathfrak{F}$ is denoted by $\mathfrak{Cm}\,\mathfrak{F}$. We can describe properties of $\mathfrak{F}$ in the corresponding (*multi*)*modal language*, having a k-ary modal $\diamond$ for each $k+1$-ary relation. *Validity* of a set Σ of such modal formulas in a relational structure $\mathfrak{F}$ (in symbols: $\mathfrak{F} \models \Sigma$) is defined as usual. Formulas of this modal language can also be considered as terms of an algebraic language, where each k-ary $\diamond$ is regarded as a k-ary function symbol. The starting point of the duality between modal logic and BAOs is the following property: for every relational structure $\mathfrak{F}$ and every modal formula φ,

$$\mathfrak{F} \models \varphi \qquad \Longleftrightarrow \qquad \mathfrak{Cm}\,\mathfrak{F} \models (\varphi = 1). \tag{4}$$

Given a class $\mathcal{C}$ of relational structures of the same signature, we denote by $\mathsf{Cm}\,\mathcal{C}$ the class of complex algebras of structures in $\mathcal{C}$, and by $\mathsf{Log}(\mathcal{C})$ the set of all modal formulas that are valid in every structure in $\mathcal{C}$. We then have the following consequence of (4): for any relational structure $\mathfrak{F}$,

$$\mathfrak{F} \models \mathsf{Log}(\mathcal{C}) \qquad \Longleftrightarrow \qquad \mathfrak{Cm}\,\mathfrak{F} \in \mathbf{H}\,\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}. \tag{5}$$

We are interested in varieties of BAOs generated by complex algebras of (special) structures (these are called *complex varieties* in [11]). The following general result will be used throughout this chapter:

THEOREM 1.1. (Goldblatt [13]) *If $\mathcal{C}$ is a class of relational structures that is closed under $\mathbb{U}\mathsf{p}$, then $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathbb{I}\,\mathcal{C}$ is a canonical variety.*

Let us have a closer look at the subdirectly irreducible algebras of these varieties.

LEMMA 1.2. ([33]) *For any class $\mathcal{C}$ of relational structures, the subdirectly irreducible members of $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathbb{I}\,\mathcal{C}$ belong to $\mathbf{S}\,\mathsf{Cm}\,\mathbb{I}_{\mathsf{p}}\,\mathcal{C}$.*

Proof. Let $\mathfrak{A} \in \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathbb{I}\,\mathcal{C}$ and let $\mathfrak{A} \rightarrowtail \prod_{i\in I} \mathfrak{A}_i$ be a subdirect embedding, for some $\mathfrak{A}_i \in \mathbf{S}\,\mathsf{Cm}\,\mathbb{I}\,\mathcal{C}$, $i \in I$. If $\mathfrak{A}$ is subdirectly irreducible then there is an $i \in I$ such that $\mathfrak{A} \cong \mathfrak{A}_i$, and so $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{Cm}\,\mathfrak{F}$ for some $\mathfrak{F} \in \mathbb{I}\,\mathcal{C}$. Then for each point x in $\mathfrak{F}$, $\mathfrak{F}^x \in \mathbb{I}_{\mathsf{p}}\,\mathbb{I}\,\mathcal{C} \subseteq \mathbb{I}_{\mathsf{p}}\,\mathcal{C}$. It is not hard to show (see e.g. [11, 3.3]) that $\mathfrak{Cm}\,\mathfrak{F} \rightarrowtail \prod_{x\in\mathfrak{F}} \mathfrak{Cm}\,\mathfrak{F}^x$ is a (subdirect) embedding. So there exist subalgebras $\mathfrak{B}_x$ of $\mathfrak{Cm}\,\mathfrak{F}^x$ such that $\mathfrak{A} \rightarrowtail \prod_{x\in\mathfrak{F}} \mathfrak{B}_x$ is a subdirect embedding as well. As $\mathfrak{A}$ is subdirectly irreducible, there is some x in $\mathfrak{F}$ such that $\mathfrak{A} \cong \mathfrak{B}_x$, and so $\mathfrak{A} \in \mathbf{S}\,\mathsf{Cm}\,\mathbb{I}_{\mathsf{p}}\,\mathcal{C}$. □

Now Theorem 1.1 and Lemma 1.2 imply the following characterisation of varieties generated by certain classes of complex algebras.

THEOREM 1.3. ([33]) *If $\mathcal{C}$ is a class of relational structures that is closed under* $\mathbb{U}\mathsf{p}$ *and* $\mathbb{I}_{\mathsf{p}}$ *, then* $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C} = \mathbf{H}\,\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}$ *is a canonical variety.*

We can also have a 'dual' structural characterisation of subdirectly irreducible algebras of these varieties. We denote by $\mathfrak{Uf}\,\mathfrak{A}$ the *ultrafilter frame* of a BAO $\mathfrak{A}$, and by $\mathfrak{Ue}\,\mathfrak{F} = \mathfrak{Uf}\,\mathfrak{Cm}\,\mathfrak{F}$ the *ultrafilter extension* of a relational structure $\mathfrak{F}$.

THEOREM 1.4. *Let $\mathcal{C}$ be a class of relational structures that is closed under* $\mathbb{U}\mathsf{p}$. *Then for every subdirectly irreducible algebra $\mathfrak{A}$,*

$\mathfrak{A} \in \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C} \iff \mathfrak{A} \in \mathbf{S}\,\mathsf{Cm}\,\mathcal{C} \iff \mathfrak{Uf}\,\mathfrak{A}$ *is a p-morphic image of some* $\mathfrak{G} \in \mathcal{C}$.

Proof. $\Leftarrow$: By Jónsson and Tarski's [26] theorem, $\mathfrak{A}$ is embeddable into $\mathfrak{Cm}\,\mathfrak{Uf}\,\mathfrak{A}$. And by duality, $\mathfrak{Cm}\,\mathfrak{Uf}\,\mathfrak{A}$ is embeddable into $\mathfrak{Cm}\,\mathfrak{G} \in \mathsf{Cm}\,\mathcal{C}$.

$\Rightarrow$: If $\mathfrak{A} \in \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}$ then there is a subdirect embedding $\mathfrak{A} \rightarrowtail \prod_{i\in I} \mathfrak{A}_i$, for some $\mathfrak{A}_i \in \mathbf{S}\,\mathsf{Cm}\,\mathcal{C}$, $i \in I$. As $\mathfrak{A}$ is subdirectly irreducible, there is an $i \in I$ such that $\mathfrak{A} \cong \mathfrak{A}_i$, that is, $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{Cm}\,\mathfrak{F}$ for some $\mathfrak{F} \in \mathcal{C}$. By duality, $\mathfrak{Uf}\,\mathfrak{A}$ is a p-morphic image of $\mathfrak{Ue}\,\mathfrak{F}$. As $\mathfrak{Ue}\,\mathfrak{F}$ is a p-morphic image of an ultrapower of $\mathfrak{F}$ (see [7, 2, 3]) and $\mathcal{C}$ is closed under taking ultraproducts, the proof is completed. □

REMARK 1.5. We can have a similar characterisation of arbitrary, not necessarily subdirectly irreducible, algebras in these kinds of varieties. If $\mathcal{C}$ is closed under $\mathbb{U}\mathsf{p}$ then

$$\mathfrak{A} \in \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C} \iff \mathfrak{Uf}\,\mathfrak{A} \text{ is a p-morphic image of } \dot{\bigcup}_{i\in I} \mathfrak{G}_i \text{ for some } \mathfrak{G}_i \in \mathcal{C}, \quad (6)$$

where $\dot{\bigcup}_{i\in I} \mathfrak{G}_i$ is the *disjoint union* of $\mathfrak{G}_i$, for $i \in I$. The proof of (6) is similar to that of Theorem 1.4, but we need to use some additional properties of the various operators such as:

- $\prod_{i\in I} \mathfrak{Cm}\,\mathfrak{G}_i \cong \mathfrak{Cm}\,\dot{\bigcup}_{i\in I} \mathfrak{G}_i$.

- [12] An ultrapower of a disjoint union of structures is a p-morphic image of a disjoint union of some ultraproducts formed from the same structures.

An example of a class $\mathcal{C}$ of relational structures that is closed under taking ultraproducts and point-generated inner substructures is the class of n-dimensional *full cylindric set algebra atom structures* (described in (2)). Disjoint unions of such structures are atom structures of *generalised cylindric set algebras*. By duality, a surjective p-morphism from such a structure onto $\mathfrak{Uf}\,\mathfrak{A}$ corresponds to a *complete representation* (see Hirsch and Hodkinson's chapter of this volume) of the *canonical embedding algebra* $\mathfrak{Cm}\,\mathfrak{Uf}\,\mathfrak{A}$ of $\mathfrak{A}$. So, as a special case of (6) one can obtain the following result of Monk. For every cylindric-type algebra $\mathfrak{A}$,

$$\mathfrak{A} \in \mathsf{RCA}_n \qquad \Longleftrightarrow \qquad \mathfrak{Cm}\,\mathfrak{Uf}\,\mathfrak{A} \text{ has a complete representation.}$$

COROLLARY 1.6. *Let $\mathcal{C}$ be a class of relational structures that is closed under $\mathbb{U}\mathsf{p}$ and $\mathbb{I}_{\mathsf{p}}$. Then for every rooted structure $\mathfrak{F}$,*

$$\mathfrak{F} \models \mathsf{Log}(\mathcal{C}) \qquad \Longleftrightarrow \qquad \mathfrak{Ue}\,\mathfrak{F} \textit{ is a p-morphic image of some } \mathfrak{G} \in \mathcal{C}.$$

Proof. By (5) and Theorem 1.3,

$$\mathfrak{F} \models \mathsf{Log}(\mathcal{C}) \qquad \Longleftrightarrow \qquad \mathfrak{Cm}\,\mathfrak{F} \in \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}.$$

As the complex algebra of a rooted structure is subdirectly irreducible [11], the statement follows from Theorem 1.4. □

As the ultrafilter extension of a finite relational structure is isomorphic to the structure itself, we obtain:

COROLLARY 1.7. *Let $\mathcal{C}$ be a class of relational structures that is closed under $\mathbb{U}\mathsf{p}$ and $\mathbb{I}_{\mathsf{p}}$. Then for every rooted finite structure $\mathfrak{F}$,*

$$\mathfrak{F} \models \mathsf{Log}(\mathcal{C}) \qquad \Longleftrightarrow \qquad \mathfrak{F} \textit{ is a p-morphic image of some } \mathfrak{G} \in \mathcal{C}.$$

2 The diagonal-free case

We would like to apply the general results of the previous section to special classes of 'n-dimensional' relational structures. To this end, for any $0 < n < \omega$, we define an *n-frame* to be a structure of the form $\langle W, T_i \rangle_{i<n}$, where W is a non-empty set and T_i is a binary relation on W, for each $i < n$. Multimodal formulas matching n-frames are called *n-modal formulas* (that is, n-modal formulas are built up from propositional variables using the Booleans and unary modal operators $\Diamond_i$ and $\Box_i$, $i < n$).

The following notion is a generalisation of atom structures of n-dimensional full diagonal-free cylindric set algebras (cf. [16, 2.7.38]).

DEFINITION 2.1. Given 1-frames $\mathfrak{F}_i = \langle W_i, R_i \rangle$, $i < n$, their *product* is the n-frame

$$\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1} = \langle W_0 \times \cdots \times W_{n-1}, \bar{R}_i \rangle_{i<n},$$

where $W_0 \times \cdots \times W_{n-1}$ is the Cartesian product of the W_i and for all $\mathbf{u}, \mathbf{v} \in W_0 \times \cdots \times W_{n-1}$ and $i < n$,

$$\mathbf{u} \bar{R}_i \mathbf{v} \quad \text{iff} \quad u_i R_i v_i \text{ and } u_j = v_j \text{ for } j \neq i,\ j < n.$$

Such n-frames we call *n-dimensional product frames.*

It is not hard to see that the product operation commutes with taking ultraproducts and point-generated inner substructures:

PROPOSITION 2.2. *Let U be an ultrafilter over some index set I, and let $\mathfrak{F}_k^i$ be a 1-frame, for $i \in I$, $k < n$. Then:*

$$\prod_{i \in I} (\mathfrak{F}_0^i \times \cdots \times \mathfrak{F}_{n-1}^i)/U \cong (\prod_{i \in I} \mathfrak{F}_0^i/U) \times \cdots \times (\prod_{i \in I} \mathfrak{F}_{n-1}^i/U).$$

PROPOSITION 2.3. *Let $\mathfrak{F} = \mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1}$ and $\mathbf{x}$ be a point in $\mathfrak{F}$. Then:*

$$\mathfrak{F}^{\mathbf{x}} = \mathfrak{F}_0^{x_0} \times \cdots \times \mathfrak{F}_{n-1}^{x_{n-1}}.$$

REMARK 2.4. Given classes $\mathcal{C}_i$ of 1-frames, for $i < n$, we can define a class $\mathcal{C}$ of n-dimensional product frames by taking

$$\mathcal{C} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1} = \{\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1} : \mathfrak{F}_i \in \mathcal{C}_i,\ i < n\}.$$

As a consequence of Propositions 2.2 and 2.3, we obtain that if each class $\mathcal{C}_i$ is defined by a set of *universal* formulas in the first-order language having one binary predicate symbol and possibly equality, then $\mathcal{C}$ is closed under taking ultraproducts and point-generated inner substructures. Here are some examples of this kind:

$$\begin{aligned}
\mathcal{C}^n_{all} &= \text{the class of all } n\text{-dimensional product frames,} \\
\mathcal{C}^n_{trans} &= \text{the class of all } n\text{-dimensional products of transitive frames,} \\
\mathcal{C}^n_{equiv} &= \text{the class of all } n\text{-dimensional products of equivalence frames,} \\
\mathcal{C}^n_{univ} &= \text{the class of all } n\text{-dimensional products of universal frames} \\
&= \text{the class of all } n\text{-dimensional full diagonal-free cylindric} \\
&\qquad\qquad \text{set algebra atom structures.}
\end{aligned}$$

By Theorem 1.3, in each of these cases $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}$ is a canonical variety. In particular, we can obtain the class RDf_n of *representable diagonal-free cylindric algebras of* *dimension* n: As $\mathbb{I}_{\mathsf{p}}\,\mathcal{C}^n_{equiv} = \mathcal{C}^n_{univ}$, $\mathsf{RDf}_n = \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^n_{univ} = \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^n_{equiv}$ holds. Moreover, by Johnson [25] (see also Halmos [14] and [15, Section 5.1]), we also have $\mathsf{RDf}_n = \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^n_{cube}$, where

$$\mathcal{C}^n_{cube} = \{\underbrace{\mathfrak{F}\times\cdots\times\mathfrak{F}}_{n} : \mathfrak{F} = \langle U, U\times U\rangle \text{ for some non-empty set } U\}$$

is yet another class of n-dimensional product frames that is closed under $\mathbb{U}\mathsf{p}$ and $\mathbb{I}_{\mathsf{p}}$.

Let us introduce notation for the corresponding n-modal logics:

$$\begin{aligned}
\mathbf{K}^n &= \mathsf{Log}(\mathcal{C}^n_{all}),\\
\mathbf{K4}^n &= \mathsf{Log}(\mathcal{C}^n_{trans}),\\
\mathbf{S5}^n &= \mathsf{Log}(\mathcal{C}^n_{equiv}) = \mathsf{Log}(\mathcal{C}^n_{univ}) = \mathsf{Log}(\mathcal{C}^n_{cube}).
\end{aligned}$$

The following theorem shows that any n-frame having n equivalence relations and being a p-morphic image of an arbitrary n-dimensional product frame is also a p-morphic image of a product of n equivalence frames.

THEOREM 2.5. ([33]) *Let $\mathfrak{F} = \langle W, T_i\rangle_{i<n}$ be an n-frame such that every T_i is an equivalence relation, for $i<n$. Suppose that $f : \mathfrak{G}_0\times\cdots\times\mathfrak{G}_{n-1} \to \mathfrak{F}$ is a surjective p-morphism, for some 1-frames $\mathfrak{G}_i = \langle U_i, R_i\rangle$, $i<n$. Then there exist 1-frames $\mathfrak{G}^*_i = \langle U_i, R^*_i\rangle$, $i<n$, such that*

- *each R^*_i is an equivalence relation extending R_i, and*
- *$f : \mathfrak{G}^*_0\times\cdots\times\mathfrak{G}^*_{n-1} \to \mathfrak{F}$ is still a surjective p-morphism.*

Proof. Let κ be an infinite cardinal $\geq \max_{i<n}|U_i|$. Then we can 'fix' the domain of f by playing the following 2-player κ-long game over f. The players $\forall$ and $\exists$ are building an increasing sequence $\langle R^\alpha_i : \alpha<\kappa\rangle$ of binary relations on U_i, for each $i<n$. At round 0, R^0_i is the reflexive closure of R_i, $i<n$. Then clearly f is still a surjective p-morphism from $\langle U_0, R^0_0\rangle\times\cdots\times\langle U_{n-1}, R^0_{n-1}\rangle$ to $\mathfrak{F}$, as each relation T_i in $\mathfrak{F}$ is reflexive.

At round $\alpha+1<\kappa$, $\forall$ picks

(i) either a tuple $\langle i,x,y\rangle$ such that $i<n$, $x,y\in U_i$, and $xR^\alpha_i y$;

(ii) or a tuple $\langle i,x,y,z\rangle$ such that $i<n$, $x,y,z\in U_i$, and $xR^\alpha_i yR^\alpha_i z$.

$\exists$ has to respond with $R^{\alpha+1}_i \supseteq R^\alpha_i$ such that f is still a surjective p-morphism from $\langle U_0, R^{\alpha+1}_0\rangle\times\cdots\times\langle U_{n-1}, R^{\alpha+1}_{n-1}\rangle$ to $\mathfrak{F}$, and either $yR^{\alpha+1}_i x$ (in case (i)), or $xR^{\alpha+1}_i z$ (in case (ii)).

At round β for limit ordinals $\beta < \kappa$, they take $R_i^\beta = \bigcup_{\alpha<\beta} R_i^\alpha$, for $i < n$. If at each round $\alpha < \kappa$ $\exists$ can respond, then $R_i^* = \bigcup_{\alpha<\kappa} R_i^\alpha$, $i < n$, would clearly be an equivalence relation as required.

Let us define a winning strategy for $\exists$. Suppose that in round α $\forall$ chooses a tuple like in (ii) (the case of (i) is similar). Then let $R_i^{\alpha+1} = R_i^\alpha \cup \{\langle x, z\rangle\}$ and $R_j^{\alpha+1} = R_j^\alpha$ for all $j < n$, $j \neq i$. We claim that f is still a surjective p-morphism from $\langle U_0, R_0^{\alpha+1}\rangle \times \cdots \times \langle U_{n-1}, R_{n-1}^{\alpha+1}\rangle$ to $\mathfrak{F}$. Indeed, the 'backward condition' clearly holds, as we added pairs only to the domain of f. As concerns f being a homomorphism, take a 'new' pair (if there is such) $\langle \mathbf{u}, \mathbf{v}\rangle$ from $\bar{R}_i^{\alpha+1}$. Then $u_i = x$ and $v_i = z$, and $u_j = v_j$ for $j < n$, $j \neq i$. Let $\mathbf{w} = \langle u_0, \ldots, u_{i-1}, y, u_{i+1}, \ldots, u_{n-1}\rangle$. Then $\mathbf{u}\bar{R}_i^\alpha \mathbf{w} \bar{R}_i^\alpha \mathbf{v}$ and, as f is p-morphism from $\langle U_0, R_0^\alpha\rangle \times \cdots \times \langle U_{n-1}, R_{n-1}^\alpha\rangle$ to $\mathfrak{F}$, $f(\mathbf{u}) T_i f(\mathbf{w}) T_i f(\mathbf{v})$. As T_i is transitive, we have $f(\mathbf{u}) T_i f(\mathbf{v})$ as required. □

Remark 2.6. Note that a similar proof would prove a stronger statement. The property of each T_i being an equivalence relation can be replaced with any property of T_i that can be defined by a set of *universal Horn* formulas in the first-order language having a binary predicate symbol and possibly equality (and there can be different such properties for different i).

Theorem 2.7. ([33]) *Let L be any canonical n-modal logic*[1] *such that* $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$*. Then* $\mathbf{S5}^n$ *is finitely axiomatisable over L:* $\mathbf{S5}^n$ *is the smallest n-modal logic containing L and the* $\mathbf{S5}$*-axioms* (3)*, for $i < n$.*

Proof. One inclusion is clear, let us prove the other. The **S5**-axioms are well-known examples of Sahlqvist formulas, and their first-order correspondent is the property of being an equivalence relation. So, by Sahlqvist's completeness theorem, the smallest n-modal logic containing L and the **S5**-axioms is canonical, and so Kripke complete. So it is enough to show that every rooted n-frame $\mathfrak{F}$ for this logic is a frame for $\mathbf{S5}^n$.

Take such an n-frame $\mathfrak{F}$. As $\mathfrak{F}$ is a frame for $\mathbf{K}^n = \mathsf{Log}(\mathcal{C}^n_{all})$, by Corollary 1.6, $\mathfrak{Ue}\,\mathfrak{F}$ is a p-morphic image of some n-dimensional product frame $\mathfrak{G}$. As $\mathfrak{F}$ validates the canonical **S5**-axioms, they also hold in $\mathfrak{Ue}\,\mathfrak{F}$, and so all the relations in $\mathfrak{Ue}\,\mathfrak{F}$ are equivalence relations. Now by Theorem 2.5, $\mathfrak{Ue}\,\mathfrak{F}$ is a p-morphic image of some $\mathfrak{G}^* \in \mathcal{C}^n_{equiv}$, and so by Corollary 1.6 again, $\mathfrak{F}$ is a frame for $\mathbf{S5}^n = \mathsf{Log}(\mathcal{C}^n_{equiv})$. □

Let us formulate a consequence of Theorems 1.3 and 2.7 in an algebraic form:

Theorem 2.8. *Let $\mathcal{C}$ be any class of n-dimensional product frames that is closed under* $\mathbb{U}\mathsf{p}$ *and* $\mathbb{I}_\mathsf{p}$*. Then the equational theory of* RDf_n *is finitely axiomatisable over the equational theory of* $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}$*: one only has to add the equations* (1)*, for $i < n$.*

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[1]By an *n-modal logic* we mean any set of n-modal formulas that contains all propositional tautologies, the formulas (K) for each $\Box_i$, and is closed under the derivation rules of Substitution, Modus Ponens and Necessitation, for $i < n$.

REMARK 2.9. By Remarks 2.4 and 2.6, we can have similar statements for any $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{K}$ in place of RDf_n, whenever $\mathcal{K} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}$ for come classes $\mathcal{C}_i$ of 1-frames, each of which is definable by Sahlqvist formulas having universal Horn first-order correspondents.

Theorems 2.7 and 2.8 show that any negative result on the equational axiomatisation of RDf_n (such as its non-finiteness [25], for $n \geq 3$) transfers to many varieties generated by complex algebras of n-dimensional product frames (or, to many-dimensional modal logics like $\mathbf{K}^n$). In other words, these theorems also mean that all the complexity of RDf_n (or its logic counterpart $\mathbf{S5}^n$) comes from the many-dimensional structure and is already present in $\mathbf{K}^n$. Though, by a general result of [9], $\mathbf{K}^n$ is known to be recursively enumerable, an axiomatisation of $\mathbf{K}^n$ should be quite complex, whenever $n \geq 3$: any such axiomatisation should contain n-modal formulas of arbitrary modal depth for each modality [29], n-modal formulas without first-order correspondents [33], and infinitely many propositional variables [32]. At the moment we cannot use Theorem 2.7 to infer the latter, as it is not known whether $\mathbf{S5}^n$ (or RDf_n) can be axiomatised using finitely many variables, whenever $n \geq 3$.

For $n = 2$, the following generalisation of $\mathsf{RDf}_2 = \mathsf{Df}_2$ (see e.g. [15, 5.1.47]) holds:

THEOREM 2.10. (Gabbay and Shehtman [9]) *Let Σ_0 and Σ_1 be sets of 1-modal formulas having universal Horn first-order correspondents, and let*

$$\mathcal{C} = \{\mathfrak{F}_0 \times \mathfrak{F}_1 : \mathfrak{F}_0 \models \Sigma_0,\ \mathfrak{F}_1 \models \Sigma_1\}.$$

Then $\mathsf{Log}(\mathcal{C})$ is the smallest 2-modal logic containing Σ_0 for $\Diamond_0$, Σ_1 for $\Diamond_1$, and the interaction axioms

$$\Diamond_0\Diamond_1 p \leftrightarrow \Diamond_1\Diamond_0 p \qquad \textit{and} \qquad \Diamond_0\Box_1 p \to \Box_1\Diamond_0 p.$$

So, in particular, $\mathbf{K}^2$, $\mathbf{K4}^2$ and $\mathbf{S5}^2$ (the logic counterpart of RDf_2) are all finitely axiomatisable. Note that in case of $\mathbf{S5}^2$ the second interaction axiom (*confluence*) follows from the first (*commutativity*).

Next, we make use of a result of Hirsch and Hodkinson [18] saying that representability of finite subdirectly irreducible *relation algebras*s undecidable. To begin with, this result implies the following:

THEOREM 2.11. ([21]) *If $n \geq 3$ then it is undecidable whether a finite subdirectly irreducible n-dimensional diagonal-free cylindric algebra is representable.*

Proof. Monk [36] introduced a construction that turns any finite subdirectly irreducible relation algebra $\mathfrak{A}$ to a finite 3-dimensional cylindric algebra $Ca_3\mathfrak{A}$ such that

- $\mathfrak{A}$ is representable as a relation algebra iff $Ca_3\mathfrak{A} \in \mathsf{RCA}_3$;
- the diagonal-free reduct $Df_3\mathfrak{A}$ of $Ca_3\mathfrak{A}$ is subdirectly irreducible and generated by 2-dimensional elements.

Now using the results of Halmos[14] and Johnson [25] (see also [15, 5.1]), we obtain that $Ca_3\mathfrak{A} \in \mathsf{RCA}_3$ iff $Df_3\mathfrak{A} \in \mathsf{RDf}_3$. Next, for every $n > 3$, we can extend $Df_3\mathfrak{A}$ to an n-dimensional diagonal-free cylindric algebra $Df_n\mathfrak{A}$ by keeping the same domain and c_0, c_1 and c_2 as in $Df_3\mathfrak{A}$, and defining c_i as the identity function, for each $3 \le i < n$. Then it is straightforward to show that $Df_3\mathfrak{A} \in \mathsf{RDf}_3$ iff $Df_n\mathfrak{A} \in \mathsf{RDf}_n$. □

Now take any finite subdirectly irreducible n-dimensional diagonal-free cylindric algebra $\mathfrak{A}$, and consider its atom structure $\mathfrak{At}_\mathfrak{A}$. Then $\mathfrak{At}_\mathfrak{A}$ is an n-frame, so by (5),

$$\mathfrak{At}_\mathfrak{A} \models \mathbf{S5}^n \qquad \Longleftrightarrow \qquad \mathfrak{A} \cong \mathfrak{Cm}\,\mathfrak{At}_\mathfrak{A} \in \mathsf{RDf}_n. \tag{7}$$

THEOREM 2.12. ([21]) *Let $n \ge 3$ and let L be any set of n-modal formulas with $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then the following hold:*

(i) *It is undecidable whether $\mathfrak{F} \models L$ for a finite rooted n-frame $\mathfrak{F}$.*

(ii) *L is not finitely axiomatisable.*

Proof. Take any finite subdirectly irreducible n-dimensional diagonal-free cylindric algebra $\mathfrak{A}$. Then its atom structure $\mathfrak{At}_\mathfrak{A}$ is rooted and all its relations are equivalence relations, so by Corollary 1.7 and Theorem 2.5,

$$\mathfrak{At}_\mathfrak{A} \models L \qquad \Longleftrightarrow \qquad \mathfrak{At}_\mathfrak{A} \models \mathbf{S5}^n.$$

Therefore, item (i) follows from (7) and Theorem 2.11.

Item (ii) clearly follows from (i), as long as a *finite axiomatisation* of L means a finitary proof system that is suitable for testing whether $\mathfrak{F} \models L$ for a finite n-frame $\mathfrak{F}$. This does not necessarily mean that only the so-called ‘orthodox’ derivation rules (Substitution, Modus Ponens and Necessitation) of modal logic are allowed. However, certain ‘non-orthodox’ rules such as some versions of the *irreflexivity rule* (see Venema’s chapter in this volume) are not suitable for this purpose. □

Observe that the atom structure $\mathfrak{At}_\mathfrak{A} = \langle W, T_i \rangle_{i<n}$ of a finite subdirectly irreducible n-dimensional diagonal-free cylindric algebra $\mathfrak{A}$ is not only rooted but, having chosen any of its points r as root, has the following property:

$$\forall x \in W\ \exists\, y_0, \dots, y_n\ \big(y_0 = r \wedge y_n = x \wedge \forall i < n\,(y_i = y_{i+1} \vee y_i T_i y_{i+1})\big). \tag{8}$$

Now, for each $w \in W$, let us introduce a propositional variable p_w, and define an n-modal formula $\varphi_{\mathfrak{A}}$ by taking

$$\Box_0^+ \dots \Box_{n-1}^+ \Big(\bigvee_{w \in W} p_w \wedge \bigwedge_{w \neq w' \in W} \neg(p_w \wedge p_{w'}) \wedge \\ \bigwedge_{\substack{i<n,\, w,w' \in W \\ wT_i w'}} p_w \to \Diamond_i p_{w'} \wedge \bigwedge_{\substack{i<n,\, w,w' \in W \\ \neg(wR_i w')}} p_w \to \neg \Diamond_i p_{w'} \Big), \quad (9)$$

where $\Box_i^+ \psi$ abbreviates $\psi \wedge \Box_i \psi$. This formula is the n-modal version of the *frame formula* (also known as *splitting formula*, see [42, 6]) of the n-frame $\mathfrak{At}_{\mathfrak{A}}$. It is clearly satisfiable in $\mathfrak{At}_{\mathfrak{A}}$, and it is supposed to describe $\mathfrak{At}_{\mathfrak{A}}$ 'up to p-morphism' in n-frames with property (8). However, as the following lemma shows, it is quite powerful in arbitrary product frames as well:

LEMMA 2.13. ([21]) *If $\varphi_{\mathfrak{A}}$ is satisfied in any n-dimensional product frame, then there is some $\mathfrak{G} \in \mathcal{C}^n_{univ}$ such that $\mathfrak{At}_{\mathfrak{A}}$ is a p-morphic image of $\mathfrak{G}$.*

Proof. Suppose $\mathfrak{M}, \mathbf{x} \models \varphi_{\mathfrak{A}}$ for some model $\mathfrak{M}$ over $\mathfrak{F} = \mathfrak{F}_0 \times \dots \times \mathfrak{F}_{n-1}$, for some $\mathfrak{F}_i = \langle U_i, R_i \rangle$, $i < n$. Take $U_i^- = \{u \in U_i : u = x_i \text{ or } x_i R_i u\}$, and define a function f from $U_0^- \times \cdots U_{n-1}^-$ to $\mathfrak{At}_{\mathfrak{A}}$ by taking,

$$f(\mathbf{u}) = w \qquad \Longleftrightarrow \qquad \mathfrak{M}, \mathbf{u} \models p_w.$$

It is not hard to show that f is well-defined and a p-morphism from the n-dimensional product $\mathfrak{G}$ of universal frames over U_i^- onto $\mathfrak{At}_{\mathfrak{A}}$. □

This lemma now implies that, for every L with $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$,

$$\neg \varphi_{\mathfrak{A}} \notin L \qquad \Longleftrightarrow \qquad \mathfrak{At}_{\mathfrak{A}} \models \mathbf{S5}^n.$$

So, by (7) and Theorem 2.11, we obtain the following generalisation of Maddux's result [34] on the undecidability of the equational theory of RDf_n, for $n \geq 3$:

THEOREM 2.14. ([21]) *Let $n \geq 3$ and let L be any set of n-modal formulas with $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then L is undecidable.*

REMARK 2.15. The undecidability of $\mathbf{S5}^n$ can also be derived from Theorem 2.12 and (7) as follows. As observed by Tarski [40], embeddability of a finite algebra $\mathfrak{A}$ can be described by an existential first-order sentence in the language of $\mathfrak{A}$. As $\mathsf{RDf}_n = \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^n_{univ}$ is a *discriminator variety*, non-embeddability of a finite diagonal-free cylindric algebra into a representable one can be described in RDf_n by an equation. Note, however, that other varieties of the form $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}$ (such as, say, $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^n_{all}$) might not be discriminator varieties and we have to use something like Lemma 2.13.

Note that one can find undecidable many-dimensional modal logics already in dimension 2. Gabelaia *et al.* [10] provide a wide choice of these logics, in a sense the most surprising among them being $\mathbf{K4}^2$. This undecidable logic is finitely axiomatisable with the natural axioms by Theorem 2.10. In the algebraic setting, we obtain that the equational theory of two commuting and confluent closure operators is undecidable.

The reader might have the impression by now that metalogical properties of, say, $\mathbf{K}^n$ and $\mathbf{S5}^n$ always go hand in hand. We mention a property for which this is not the case: while $\mathbf{K}^n$ does have the *finite model property* [9], $\mathbf{S5}^n$ does not [30], whenever $n \geq 3$.

3 With diagonals

We define an *$n\delta$-frame* to be a structure of the form $\langle W, T_i, E_{ij}\rangle_{i,j<n}$, where $\langle W, T_i\rangle_{i<n}$ is an n-frame and E_{ij} is a subset of W, for $i, j < n$. Multimodal formulas matching $n\delta$-frames, *$n\delta$-modal formulas*, are built up from propositional variables using the Booleans, unary modal operators $\Diamond_i$ and $\Box_i$, and constants δ_{ij}, for $i, j < n$.

The following notion is a generalisation of atom structures of n-dimensional full cylindric set algebras.

Definition 3.1. Given 1-frames $\mathfrak{F}_i = \langle W_i, R_i\rangle$, $i < n$, their *δ-product* is the $n\delta$-frame

$$(\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1})^\delta = \langle W_0 \times \cdots \times W_{n-1}, \bar{R}_i, Id_{ij}\rangle_{i,j<n},$$

where $\langle W_0 \times \cdots \times W_{n-1}, \bar{R}_i\rangle_{i<n} = \mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1}$ and, for $i, j < n$,

$$Id_{ij} = \{\mathbf{u} \in W_0 \times \cdots \times W_{n-1} \,:\, u_i = u_j\}.$$

Such n-frames we call *n-dimensional δ-product frames.*

We have the analogues of Propositions 2.2 and 2.3:

Proposition 3.2. *Let U be an ultrafilter over some index set I, and let $\mathfrak{F}^i_k$ be a 1-frame, for $i \in I$, $k < n$. Then:*

$$\prod_{i\in I}(\mathfrak{F}^i_0 \times \cdots \times \mathfrak{F}^i_{n-1})^\delta/U \;\cong\; \big((\prod_{i\in I}\mathfrak{F}^i_0/U) \times \cdots \times (\prod_{i\in I}\mathfrak{F}^i_{n-1}/U)\big)^\delta.$$

Proposition 3.3. *Let $\mathfrak{F} = (\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1})^\delta$ and $\mathbf{x}$ be a point in $\mathfrak{F}$. Then:*

$$\mathfrak{F}^{\mathbf{x}} = (\mathfrak{F}_0^{x_0} \times \cdots \times \mathfrak{F}_{n-1}^{x_{n-1}})^\delta.$$

Given a class $\mathcal{C}$ of n-dimensional product frames, we denote by $\mathcal{C}^\delta$ the corresponding class of δ-product frames:

$$\mathcal{C}^\delta = \{(\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1})^\delta : \mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1} \in \mathcal{C}\}.$$

It is straightforward to see the following:

Proposition 3.4. *For any class $\mathcal{C}$ of n-dimensional product frames,* $\mathsf{Log}(\mathcal{C}^\delta)$ *is a conservative extension of* $\mathsf{Log}(\mathcal{C})$.

Remark 3.5. Just like in Remark 2.4, observe that as a consequence of Propositions 3.2 and 3.3 we obtain the following. If each class $\mathcal{C}_i$ of 1-frames, for $i < n$, is defined by *universal* formulas in the first-order language having one binary predicate symbol and possibly equality, then $(\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1})^\delta$ is closed under taking ultraproducts and point-generated inner substructures. Here are some examples of this kind:

$$\begin{aligned}
\mathcal{C}^{n\delta}_{all} &= \text{the class of all } n\text{-dimensional } \delta\text{-product frames},\\
\mathcal{C}^{n\delta}_{equiv} &= \text{the class of all } n\text{-dimensional } \delta\text{-products of equivalence frames},\\
\mathcal{C}^{n\delta}_{univ} &= \text{the class of all } n\text{-dimensional } \delta\text{-products of universal frames}.
\end{aligned}$$

Since $\mathcal{C}$ is closed under $\mathbb{U}\mathsf{p}$ and $\mathbb{I}_\mathsf{p}$ in each of these cases, $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}$ is a canonical variety by Theorem 1.3. As $\mathbb{I}_\mathsf{p}\,\mathcal{C}^{n\delta}_{equiv} = \mathcal{C}^{n\delta}_{univ}$, we have $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^{n\delta}_{univ} = \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^{n\delta}_{equiv}$.

However, consider now the class

$$\begin{aligned}
\mathcal{C}^{n\delta}_{cube} &= \{(\underbrace{\mathfrak{F} \times \cdots \times \mathfrak{F}}_{n})^\delta : \mathfrak{F} = \langle U, U \times U\rangle \text{ for some non-empty set } U\}\\
&= \text{the class of all } n\text{-dimensional full cylindric set algebra}\\
&\qquad\qquad\qquad\qquad \text{atom structures}
\end{aligned}$$

that is also closed under $\mathbb{U}\mathsf{p}$ and $\mathbb{I}_\mathsf{p}$. Unlike in the diagonal-free case, $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^{n\delta}_{cube} = \mathsf{RCA}_n$ is properly contained in $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^{n\delta}_{equiv}$, as for instance the equations $c_i d_{ij} = 1$ fail in the latter. (As we shall see below, in a sense they are the only missing ones.)

Let us introduce notation for some $n\delta$-modal logics:

$$\begin{aligned}
\mathbf{K}^{n\delta} &= \mathsf{Log}(\mathcal{C}^{n\delta}_{all})\\
\mathbf{S5}^{n\delta} &= \mathsf{Log}(\mathcal{C}^{n\delta}_{equiv}) = \mathsf{Log}(\mathcal{C}^{n\delta}_{univ}).
\end{aligned}$$

The proof of the following two theorems are completely analogous to the respective proofs of Theorems 2.5 and 2.7:

THEOREM 3.6. *Let $\mathfrak{F} = \langle W, T_i, E_{ij} \rangle_{i,j<n}$ be an $n\delta$-frame such that every T_i is an equivalence relation, for $i < n$. Suppose that $f : (\mathfrak{G}_0 \times \cdots \times \mathfrak{G}_{n-1})^\delta \to \mathfrak{F}$ is a surjective p-morphism, for some 1-frames $\mathfrak{G}_i = \langle U_i, R_i \rangle$, $i < n$. Then there exist 1-frames $\mathfrak{G}_i^* = \langle U_i, R_i^* \rangle$, $i < n$, such that*

- *each R_i^* is an equivalence relation extending R_i, and*
- *$f : (\mathfrak{G}_0^* \times \cdots \times \mathfrak{G}_{n-1}^*)^\delta \to \mathfrak{F}$ is still a surjective p-morphism.*

THEOREM 3.7. *Let L be any canonical $n\delta$-modal logic with $\mathbf{K}^{n\delta} \subseteq L \subseteq \mathbf{S5}^{n\delta}$. Then $\mathbf{S5}^{n\delta}$ is finitely axiomatisable over L: $\mathbf{S5}^{n\delta}$ is the smallest $n\delta$-modal logic containing L and the $\mathbf{S5}$-axioms (3), for $i < n$.*

It turns out that the equations $c_i d_{ij} = 1$ (or, the $n\delta$-modal formulas $\Diamond_i \delta_{ij}$) are quite strong in the sense that they can 'force' the **S5**-properties in the presence of 'many-dimensionality', as the following surprising theorems show:

THEOREM 3.8. *Let $\mathfrak{F} = \langle W, T_i, E_{ij} \rangle_{i,j<n}$ be an $n\delta$-frame such that, for all $i, j < n$,*

$$\text{for all } w \in W \text{ there is some } w' \in E_{ij} \text{ with } w T_i w'. \tag{10}$$

Suppose that $f : (\mathfrak{G}_0 \times \cdots \times \mathfrak{G}_{n-1})^\delta \to \mathfrak{F}$ is a surjective p-morphism, for some 1-frames $\mathfrak{G}_i = \langle U_i, R_i \rangle$, $i < n$. Then $U_i = U_j$ and R_i is the universal relation on U_i, for all $i, j < n$.

Proof. We show first that $U_i \subseteq U_j$, for any $i, j < n$, $i \neq j$. To this end, let $u \in U_i$ and take any $\mathbf{x} \in U_0 \times \cdots \times U_{n-1}$ such that $x_i = u$. By (10), there is some $w \in E_{ij}$ such that $f(\mathbf{x}) T_j w$. As f is a p-morphism, there is $\mathbf{y} \in Id_{ij}$ such that $\mathbf{x} \bar{R}_j \mathbf{y}$, so $u = x_i = y_i = y_j \in U_j$ as required.

Next, we show that $u R_i u'$ hold, for all $i < n$, $u, u' \in U_i$. To this end, take some $j \neq i$ and $\mathbf{x} \in {}^n U_i$ such that $x_i = u$ and $x_j = u'$. As f is a p-morphism, there is $\mathbf{y} \in Id_{ij}$ such that $\mathbf{x} \bar{R}_i \mathbf{y}$, so $u = x_i R_i y_i = y_j = x_j = u'$. □

THEOREM 3.9. *Let L be any canonical $n\delta$-modal logic with $\mathbf{K}^{n\delta} \subseteq L \subseteq \mathsf{Log}(\mathcal{C}^{n\delta}_{cube})$. Then $\mathsf{Log}(\mathcal{C}^{n\delta}_{cube})$ is finitely axiomatisable over L: $\mathsf{Log}(\mathcal{C}^{n\delta}_{cube})$ is the smallest $n\delta$-modal logic containing L and the $n\delta$-formulas $\Diamond_i \delta_{ij}$, for $i, j < n$.*

Proof. Like that of Theorem 2.7, using Theorem 3.8 and that each $\Diamond_i \delta_{ij}$ is a Sahlqvist formula, with property (10) being its first-order correspondent. □

A consequence of Theorems 1.3 and 3.9 formulated in an algebraic setting is as follows:

THEOREM 3.10. *Let $\mathcal{C}$ be any class of n-dimensional δ-product frames that is closed under $\mathbb{U}\mathsf{p}$ and $\mathbb{I}_\mathsf{p}$. Then the equational theory of RCA_n is finitely axiomatisable over the equational theory of $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}$: one only has to add the equations $c_i d_{ij} = 1$, for $i, j < n$.*

Theorems 3.9 and 3.10 show that any negative result on the equational axiomatisation of RCA_n transfers to many varieties generated by complex algebras of n-dimensional δ-product frames (or, to many-dimensional modal logics like $\mathbf{K}^{n\delta}$). There are many such, whenever $n \geq 3$: $\mathsf{RCA}_n = \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^{n\delta}_{cube}$ is not only not finitely axiomatisable [37], but cannot be axiomatised using finitely many variables and finitely many occurrences of the diagonals [1]. RCA_n cannot be axiomatised using only Sahlqvist equations [22, 41]. Moreover, it is also known [20] that the class of all $n\delta$-frames $\mathfrak{F}$ such that $\mathfrak{F} \models \mathsf{Log}(\mathcal{C}^{n\delta}_{cube})$ (*strongly representable cylindric atom structures*) is not closed under ultraproducts, so $\mathsf{Log}(\mathcal{C}^{n\delta}_{cube})$ cannot be axiomatised by any set of $n\delta$-modal formulas having first-order correspondents. The even more general result of [24] saying that *representable relation algebras* do not have a canonical axiomatisation might also hold for RCA_n. So all these are true for, say, $\mathsf{Log}(\mathcal{C}^{n\delta}_{all}) = \mathbf{K}^{n\delta}$. Though it is not hard to see that $\mathbf{K}^{n\delta}$ is recursively enumerable, there is a further difficulty in finding an explicit infinite axiomatisation for it. The known explicit (infinite) equational axiomatisations for RCA_n (for $n \geq 3$) [37, 16, 17] (see also [15, 4.1] and [19, 8.3]) all make use of RCA_n being a discriminator variety. But the algebraic counterpart $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^{n\delta}_{all}$ of $\mathbf{K}^{n\delta}$ is not such.

Remark 3.11. Moreover, when we have diagonals, finding an axiomatisation can be tricky even for $n = 2$. Though RCA_2 is known to be finitely axiomatisable (see e.g. [15, 3.2.65]), and it is also finitely axiomatisable over $\mathbf{K}^{2\delta}$ by Theorem 3.10, somewhat surprisingly $\mathbf{K}^{2\delta}$ is not even axiomatisable using finitely many variables, as shown by Kikot [27].

Let us next turn to decision problems. Hodkinson [23] shows that it is undecidable whether a finite subdirectly irreducible n-dimensional cylindric algebra is representable, for any finite $n \geq 3$. Using this, with diagonals we can have a bit better than Theorem 2.12:

Theorem 3.12. *Let $n \geq 3$ and let L be any set of $n\delta$-modal formulas with $\mathbf{K}^{n\delta} \subseteq L \subseteq \mathsf{Log}(\mathcal{C}^{n\delta}_{cube})$. Then the following hold:*

(i) *It is undecidable whether $\mathfrak{F} \models L$ for a finite rooted $n\delta$-frame $\mathfrak{F}$.*

(ii) *L is not finitely axiomatisable.*

Proof. Like that of Theorem 2.12, using Theorem 3.8. □

Observe that the atom structure $\mathfrak{At}_{\mathfrak{A}} = \langle W, T_i, E_{ij}\rangle_{i,j<n}$ of a finite subdirectly irreducible n-dimensional cylindric algebra $\mathfrak{A}$ not only has property (8), but it also has (10). Now define an $n\delta$-modal formula $\psi_{\mathfrak{A}}$ by adding the following conjunct to $\varphi_{\mathfrak{A}}$ in (9):

$$\Box_0^+ \dots \Box_{n-1}^+ \bigwedge_{i,j<n} \big(\delta_{ij} \leftrightarrow \bigvee_{w \in E_{ij}} p_w\big). \tag{11}$$

Then $\psi_{\mathfrak{A}}$ is satisfied in $\mathfrak{At}_{\mathfrak{A}}$, and we have the following analogue of Lemma 2.13:

LEMMA 3.13. *If $\psi_{\mathfrak{A}}$ is satisfied in any n-dimensional δ-product frame, then there is some $\mathfrak{G}^\delta \in \mathcal{C}^{n\delta}_{cube}$ such that $\mathfrak{At}_{\mathfrak{A}}$ is a p-morphic image of $\mathfrak{G}^\delta$.*

Proof. Suppose that $\psi_{\mathfrak{A}}$ is satisfied in $\mathfrak{F}^\delta$ for some n-dimensional product frame $\mathfrak{F}$. Then $\varphi_{\mathfrak{A}}$ is satisfied in $\mathfrak{F}$. Now define $\mathfrak{G} \in \mathcal{C}^n_{univ}$ and f as in the proof of Lemma 2.13. As $\varphi_{\mathfrak{A}}$ is satisfied in $\mathfrak{F}$, f is a p-morphism from $\mathfrak{G}$ onto the diagonal-free reduct of $\mathfrak{At}_{\mathfrak{A}}$. However, by (11), f is in fact a p-morphism from $\mathfrak{G}^\delta$ onto $\mathfrak{At}_{\mathfrak{A}}$. As $\mathfrak{At}_{\mathfrak{A}}$ has property (10), Theorem 3.8 implies that $\mathfrak{G}^\delta \in \mathcal{C}^{n\delta}_{cube}$, as required. □

Now we can have the analogue of Theorem 2.14:

THEOREM 3.14. *Let $n \geq 3$ and let L be any set of $n\delta$-modal formulas with $\mathbf{K}^{n\delta} \subseteq L \subseteq \mathsf{Log}(\mathcal{C}^{n\delta}_{cube})$. Then L is undecidable.*

Proof. Like that of Theorem 2.14, using Lemma 3.13. Note that if $\mathbf{K}^{n\delta} \subseteq \mathsf{Log}(\mathcal{C}) \subseteq \mathbf{S5}^{n\delta}$ for some class $\mathcal{C}$ of n-dimensional δ-product frames, then the undecidability of $\mathsf{Log}(\mathcal{C})$ already follows from Theorem 2.14 and Proposition 3.4. □

REMARK 3.15. There are cases when adding the diagonal does matter in the decision problem. An example is $\mathbf{K}^2$ that is decidable [9], while $\mathbf{K}^{2\delta}$ is not [28].

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