Products of modal logics with diagonal constant lacking the finite model property

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Abstract. Two-dimensional products of modal logics having at least one 'non-transitive' component, such as $\mathbf{K} \times \mathbf{K}$, $\mathbf{K} \times \mathbf{K4}$, and $\mathbf{K} \times \mathbf{S5}$, are often known to be decidable and have the finite model property. Here we show that by adding the diagonal constant to the language this might change: one can have formulas that are only satisfiable in infinite 'abstract' models for these logics.

1 Introduction

The formation of Cartesian products of various structures (vector and topological spaces, algebras, etc.) is a standard mathematical way of modelling the multi-dimensional character of our world. In modal logic, products of Kripke frames are natural constructions allowing us to reflect interactions between modal operators representing time, space, knowledge, actions, etc. The product construction as a combination method on modal logics has been used in applications in computer science and artificial intelligence (see [2, 10] and references therein) ever since its introduction in the 1970s [15, 16].

If the component frames are unimodal, then the modal language speaking about product frames has two interacting box operators (and the corresponding diamonds), one coming from each component. Product logics are Kripke complete logics in this language, determined by classes of product frames. In general they can also have other, non-product, frames. So one can consider two different kinds of the *finite model property* (fmp): one w.r.t. arbitrary ('abstract' or 'non-standard') frames, and a stronger one, w.r.t. product frames only. As the fmp can be an important tool in establishing decidability of a modal logic, it has been extensively studied in connection with product logics as well. In particular, if one of the component logics is either $\mathbf{K}$ or $\mathbf{S5}$, then many product logics are known to be decidable and have the fmp [3, 2, 17]. Product logics like $\mathbf{K} \times \mathbf{K}$, $\mathbf{K} \times \mathbf{S5}$ and $\mathbf{S5} \times \mathbf{S5}$ even have the fmp w.r.t. product frames [4, 2, 13]. On the other hand, when both component logics have transitive frames only, such as $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{GL}$, then product logics are usually undecidable and lack the 'abstract' fmp [5, 6, 14].

Here we take the first steps in investigating the expressive power of extensions of decidable product logics with 'dimension-connecting' connectives. Perhaps the simplest and most natural operation of this sort is the *diagonal constant* δ. The main reason for introducing such a constant has been to give a 'modal treatment' of equality of classical first-order logic. Modal algebras for the product logic $\mathbf{S5} \times \mathbf{S5}$ extended with diagonal constant are called *representable cylindric algebras of dimension 2* and have been extensively studied in the algebraic logic literature [8]. '$\mathbf{S5} \times \mathbf{S5}$ plus δ' is known to be decidable and has the finite model property, even w.r.t. product frames [13]. Adding the diagonal constant does not even change the NExpTime-completeness of the $\mathbf{S5} \times \mathbf{S5}$-satisfiability problem [7, 12]. Here we show that, rather surprisingly, products of other decidable modal logics may behave differently (see Theorem 3 below). Though, say, $\mathbf{K} \times \mathbf{S5}$ also has a NExpTime-complete satisfiability problem and has the fmp w.r.t. product frames [3, 12, 2], by adding the diagonal constant to the language one can find some formula such that *any* frame for $\mathbf{K} \times \mathbf{S5}$ satisfying it must be infinite.

As concerns our formulas and the technique used in the proofs below, we would like to emphasise that—unlike [14, 5, 6, 2]—in general here we are *neither* dealing with transitive frames, *nor* having some kind of universal modality in the language. So the fact that infinity can be forced by a formula is quite unusual. We hope that the formulas below will either help in encoding some undecidable problem and showing that decidable product logics like $\mathbf{K} \times \mathbf{K}$, $\mathbf{K} \times \mathbf{S5}$ and $\mathbf{K} \times \mathbf{K4}$ become undecidable if we add the diagonal constant, or give some hints on how their infinite models can be represented by some finite means, say, using mosaics or loop-controlled tableaux, in order to prove decidability.

2 Products and δ-products

We assume as known the fundamental notions of modal logic (such as uni- and multimodal Kripke frames and models, satisfiability and validity of formulas, generated subframes, etc.) and their basic properties, and use a standard notation.

Let us begin with introducing the product construction and its extension with a diagonal element. Given unimodal Kripke frames $\mathfrak{F}_0 = (W_0, R_0)$ and $\mathfrak{F}_1 = (W_1, R_1)$, their *product* is the bimodal frame

$$\mathfrak{F}_0 \times \mathfrak{F}_1 = (W_0 \times W_1, R_h, R_v),$$

where $W_0 \times W_1$ is the Cartesian product of sets W_0 and W_1 and the binary relations R_h and R_v are defined by taking, for all $u, u' \in W_0$, $v, v' \in W_1$,

$$(u,v)R_h(u',v') \quad \text{iff} \quad uR_0u' \text{ and } v = v',$$
$$(u,v)R_v(u',v') \quad \text{iff} \quad vR_1v' \text{ and } u = u'.$$

The *δ-product* of $\mathfrak{F}_0$ and $\mathfrak{F}_1$ is the 3-modal frame

$$\mathfrak{F}_0 \times^\delta \mathfrak{F}_1 = (W_0 \times W_1, R_h, R_v, D),$$

where $(W_0 \times W_1, R_h, R_v) = \mathfrak{F}_0 \times \mathfrak{F}_1$ and

$$D = \{(u,u) : u \in W_0 \cap W_1\}.$$

The respective modal languages speaking about product and δ-product frames are defined as follows.

$$\mathcal{L}_2: \qquad \psi = p \mid \top \mid \bot \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \Diamond_0\psi \mid \Box_0\psi \mid \Diamond_1\psi \mid \Box_1\psi$$
$$\mathcal{L}_2^\delta: \qquad \psi = p \mid \top \mid \bot \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \Diamond_0\psi \mid \Box_0\psi \mid \Diamond_1\psi \mid \Box_1\psi \mid \delta$$

For $i = 0, 1$, let L_i be a Kripke complete unimodal logic in the language using the modal operators $\Diamond_i$ and $\Box_i$. The *product* and *δ-product* of L_0 and L_1 are defined, respectively, as

$$L_0 \times L_1 = \{\psi \in \mathcal{L}_2 : \psi \text{ is valid in } \mathfrak{F}_0 \times \mathfrak{F}_1,\ \mathfrak{F}_i \text{ is a frame for } L_i,\ i = 0, 1\}$$
$$L_0 \times^\delta L_1 = \{\psi \in \mathcal{L}_2^\delta : \psi \text{ is valid in } \mathfrak{F}_0 \times^\delta \mathfrak{F}_1,\ \mathfrak{F}_i \text{ is a frame for } L_i,\ i = 0, 1\}.$$

The following proposition shows that we can define δ-products differently, in a way that might look more natural to some:

Proposition 1. *For all Kripke complete logics L_0 and L_1,*

$$L_0 \times^\delta L_1 = \{\psi \in \mathcal{L}_2^\delta : \psi \textit{ is valid in } \mathfrak{F}_0 \times^\delta \mathfrak{F}_1,\ \mathfrak{F}_i \textit{ is a frame for } L_i,\ i = 0, 1,\ \mathfrak{F}_0 \textit{ and } \mathfrak{F}_1 \textit{ have the same set of worlds}\}.$$

Proof. One inclusion is obvious. For the other, suppose $\psi \notin L_0 \times^\delta L_1$, that is, $\mathfrak{F}_0 \times^\delta \mathfrak{F}_1 \not\models \psi$, where $\mathfrak{F}_i = (W_i, R_i)$ is a frame for L_i, $i = 0, 1$. We define frames $\mathfrak{G}_0$ and $\mathfrak{G}_1$ such that:

1. $\mathfrak{G}_i$ is a frame for L_i, for $i = 0, 1$.
2. $\mathfrak{G}_0$ and $\mathfrak{G}_1$ have the same set of worlds, $W_0 \cup W_1$.
3. $\mathfrak{F}_0 \times^\delta \mathfrak{F}_1$ is a generated subframe of $\mathfrak{G}_0 \times^\delta \mathfrak{G}_1$.

To this end, for $i = 0, 1$, let $\mathfrak{H}_i$ be either the one-element irreflexive or the one-element reflexive frame, depending on which of these two is a frame for L_i. (As a Kripke complete logic is always consistent, by Makinson's theorem [11] at least one of them would do for sure.) Now, for $\{i, j\} = \{0, 1\}$, let $\mathfrak{G}_i$ be the disjoint union of $\mathfrak{F}_i$ and $\big(W_j - (W_0 \cap W_1)\big)$-many copies of $\mathfrak{H}_i$.

The proof of the following statement is straightforward from the definitions:

Proposition 2. *$L_0 \times^\delta L_1$ is a conservative extension of $L_0 \times L_1$.*

At first sight, the diagonal constant can only be meaningfully used in applications where the domains of the two component frames consist of objects of similar kinds, or at least overlap. However, as modal languages cannot distinguish between isomorphic frames, in fact *any* subset $D \subseteq W_0 \times W_1$ can be considered as an interpretation of the diagonal constant, once it has the following properties:

$$\forall x \in W_0,\ \forall y, y' \in W_1\ \big((x, y), (x, y') \in D \implies y = y'\big),$$
$$\forall x, x' \in W_0,\ \forall y \in W_1\ \big((x, y), (x', y) \in D \implies x = x'\big).$$

So, say, when a product frame represents the movement of some objects in time, then the diagonal constant can be used for collecting a set of special time-stamped objects, provided no special object is chosen twice and at every moment of time at most one special object is chosen.

3 Main results

Although δ-product logics are determined by classes of δ-product frames, of course there are other, non-δ-product, frames for them. As usual, a multimodal logic L (in particular, a δ-product logic $L_0 \times^\delta L_1$) is said to have the (*abstract*) *finite model property* (*fmp*, for short) if, for every formula φ in the language of L, if $\varphi \notin L$ then there is a finite frame $\mathfrak{F}$ for L such that $\mathfrak{F} \not\models \varphi$. (By a standard argument, this means that $\mathfrak{M} \not\models \varphi$ for some *finite model* $\mathfrak{M}$ for L; see, e.g., [1].) Below we show that many δ-product logics (with $\mathbf{K} \times^\delta \mathbf{K}$, $\mathbf{K} \times^\delta \mathbf{K4}$ and $\mathbf{K} \times^\delta \mathbf{S5}$ among them) lack the fmp.

Let φ_∞ be the conjunction of the following three (variable-free) $\mathcal{L}_2^\delta$-formulas:

$$\Diamond_1 \Diamond_0 (\delta \wedge \Box_0 \bot) \tag{1}$$

$$\Box_1 \Diamond_0 (\neg\delta \wedge \Diamond_0 \top \wedge \Box_0 \delta) \tag{2}$$

$$\Box_0 \Diamond_1 \delta \tag{3}$$

First we show that *any* frame for δ-product logics satisfying φ_∞ should be *infinite*. To this end, we have a closer look at arbitrary (not necessarily δ-product) frames for δ-product logics. It is straightforward to see that δ-product frames always have the following first-order properties:

$$\forall xyz\,\big(xR_v y \wedge yR_h z \to \exists u\,(xR_h u \wedge uR_v z)\big)$$
$$\forall xyz\,\big(xR_h y \wedge yR_v z \to \exists u\,(xR_v u \wedge uR_h z)\big)$$
$$\forall xyz\,\big(xR_v y \wedge xR_h z \to \exists u\,(yR_h u \wedge zR_v u)\big)$$
$$\forall xyzu\,\big(xR_h y \wedge xR_h z \wedge zR_h u \wedge D(y) \wedge D(u) \to y = u\big)$$

These properties are modally definable by the respective $\mathcal{L}_2^\delta$-formulas:

(lcom) $\quad \Diamond_1 \Diamond_0 p \to \Diamond_0 \Diamond_1 p$

(rcom) $\quad \Diamond_0 \Diamond_1 p \to \Diamond_1 \Diamond_0 p$

(chr) $\quad \Diamond_0 \Box_1 p \to \Box_1 \Diamond_0 p$

(diag) $\quad \Diamond_0 (\delta \wedge p) \to \Box_0 \Box_0 (\delta \to p)$

Theorem 1. *Let* $\mathfrak{F} = (W, R_0, R_1, \Delta)$ *be any frame validating* (lcom), (rcom), (chr) *and* (diag). *If* φ_∞ *is satisfied in* $\mathfrak{F}$, *then* $\mathfrak{F}$ *should be infinite.*

Proof. Let $\mathfrak{F} = (W, R_0, R_1, \Delta)$ be as required and suppose that φ_∞ is satisfied at point r of a model $\mathfrak{M}$ based on $\mathfrak{F}$. We define inductively four infinite sequences

$$x_0, x_1, x_2, \ldots, \quad y_0, y_1, y_2, \ldots, \quad u_0, u_1, u_2, \ldots \quad \text{and} \quad v_0, v_1, v_2, \ldots$$

of points from W such that, for every $i < \omega$,

(gen1) $\quad (\mathfrak{M}, x_i) \models \delta$,

(gen2) $\quad (\mathfrak{M}, y_i) \models \neg\delta \wedge \Diamond_0 \top \wedge \Box_0 \delta$

(gen3) rR_1u_i, $u_iR_0x_i$, $u_iR_0y_i$, and $y_iR_0x_i$

(gen4) if $i > 0$ then rR_0v_i, $v_iR_1x_i$ and $v_iR_1y_{i-1}$,

see Fig. 1. (We do not claim at this point that, say, all the x_i are distinct.)

To begin with, by (1), there are u_0, x_0 such that $rR_1u_0R_0x_0$ and

$$(\mathfrak{M}, x_0) \models \delta \wedge \Box_0\bot. \tag{4}$$

By (2), there is y_0 such that $u_0R_0y_0$ and $(\mathfrak{M}, y_0) \models \neg\delta \wedge \Diamond_0\top \wedge \Box_0\delta$. By (diag), we have that $y_0R_0x_0$, and so **(gen1)**–**(gen3)** hold for $i = 0$.

Now suppose that, for some $n < \omega$, x_i and y_i with **(gen1)**–**(gen4)** have already been defined for all $i \leq n$. By **(gen3)** for $i = n$ and by (lcom), there is v_{n+1} such that $rR_0v_{n+1}R_1y_n$. So by (3), there is x_{n+1} such that $(\mathfrak{M}, x_{n+1}) \models \delta$ and $v_{n+1}R_1x_{n+1}$. Now by (rcom), there is u_{n+1} such that $rR_1u_{n+1}R_0x_{n+1}$. So, by (2), there is y_{n+1} such that $u_{n+1}R_0y_{n+1}$ and $(\mathfrak{M}, y_{n+1}) \models \neg\delta \wedge \Diamond_0\top \wedge \Box_0\delta$, and so by (diag) $y_{n+1}R_0x_{n+1}$, as required (see Fig. 1).

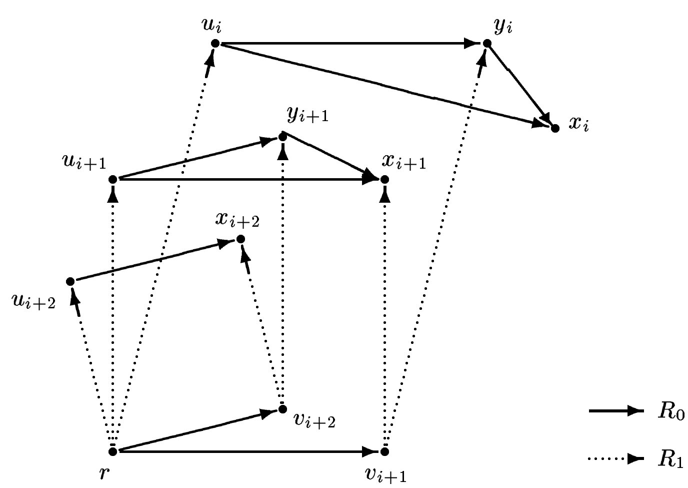

Fig. 1. Generating the points x_i, y_i, u_i and v_i.

Now we will prove that in fact all the x_n are different. We claim that, for all $n < \omega$,

$$(\mathfrak{M}, x_n) \models \Diamond_0^n\top \wedge \Box_0^{n+1}\bot \quad \text{and} \quad (\mathfrak{M}, y_n) \models \Diamond_0^{n+1}\top \wedge \Box_0^{n+2}\bot \tag{5}$$

(here $\Diamond_0^0\top = \top$, $\Diamond_0^{n+1}\top = \Diamond_0\Diamond_0^n\top$, $\Box_0^0\bot = \bot$ and $\Box_0^{n+1}\bot = \Box_0\Box_0^n\bot$). In order to prove this claim, first observe that, by **(gen1)**–**(gen3)** and (diag), we have

$$\forall z\,(y_iR_0z \to z = x_i), \tag{6}$$

for all $i < \omega$. Now we prove (5) by induction on n. For $n = 0$: Obviously, we have $(\mathfrak{M}, x_0) \models \top$, and $(\mathfrak{M}, x_0) \models \Box_0\bot$ by (4). $(\mathfrak{M}, y_0) \models \Diamond_0\top$ by **(gen2)**, and $(\mathfrak{M}, y_0) \models \Box_0^2\bot$ by (6) and $(\mathfrak{M}, x_0) \models \Box_0\bot$.

Now suppose that (5) holds for n. As $(\mathfrak{M}, y_n) \models \Diamond_0^{n+1}\top$, by (lcom), (chr) and **(gen4)** we have $(\mathfrak{M}, x_{n+1}) \models \Diamond_0^{n+1}\top$. Therefore, we obtain $(\mathfrak{M}, y_{n+1}) \models \Diamond_0^{n+2}\top$ by **(gen3)**. As $(\mathfrak{M}, y_n) \models \Box_0^{n+2}\bot$, we have $(\mathfrak{M}, x_{n+1}) \models \Box_0^{n+2}\bot$ by (lcom), (chr) and **(gen4)**. Therefore, by (6), we have $(\mathfrak{M}, y_{n+1}) \models \Box_0^{n+3}\bot$, as required.

Next, we consider a variant of φ_∞ that is satisfiable in δ-product frames with a reflexive second component. (The idea is a version of the 'chessboard-trick' of [6].) We introduce a fresh propositional variable v, and define φ_∞^r as the conjunction of the following $\mathcal{L}_2^\delta$-formulas:

$$\begin{aligned}
&\Box_1(\Diamond_0 v \to v)\\
&\Diamond_1\big(v \land \Diamond_0(\delta \land \Box_0\bot)\big)\\
&\Box_1\big(v \to \Diamond_0(\neg\delta \land \Diamond_0\top \land \Box_0\delta)\big)\\
&\Box_0\Diamond_1(v \land \delta)
\end{aligned}$$

Theorem 2. *Let* $\mathfrak{F} = (W, R_0, R_1, \Delta)$ *be any frame validating* (lcom), (rcom), (chr) *and* (diag). *If* φ_∞^r *is satisfied in* $\mathfrak{F}$, *then* $\mathfrak{F}$ *should be infinite.*

Proof. It is analogous to the proof of Theorem 1. Observe that if $(\mathfrak{M}, r) \models \varphi_\infty^r$ then $(\mathfrak{M}, u_i) \models v$, for all $i < \omega$.

Next, we show that φ_∞ and φ_∞^r are satisfiable in certain infinite δ-product frames. Indeed, define binary relations on $\omega + 1$ by taking

$$\begin{aligned}
R_0 &= \{(\omega, n) : n < \omega\} \cup \{(n+1, n) : n < \omega\},\\
R_1 &= \{(\omega, n) : n < \omega\},\\
R_1^{refl} &= \text{reflexive closure of } R_1,\\
R_1^{univ} &= \text{the universal relation on } \omega + 1,
\end{aligned}$$

and let

$$\begin{aligned}
\mathfrak{H}_0 &= (\omega + 1, R_0),\\
\mathfrak{H}_1 &= (\omega + 1, R_1),\\
\mathfrak{H}_1^{refl} &= (\omega + 1, R_1^{refl}),\\
\mathfrak{H}_1^{univ} &= (\omega + 1, R_1^{univ}).
\end{aligned}$$

Theorem 3. *Let* L_0 *be any Kripke complete unimodal logic such that* $\mathfrak{H}_0$ *is a frame for* L_0 *(such as, e.g.,* **K***) and let* L_1 *be any Kripke complete unimodal logic such that either* $\mathfrak{H}_1$ *or* $\mathfrak{H}_1^{refl}$ *or* $\mathfrak{H}_1^{univ}$ *is a frame for* L_1 *(such as, e.g.,* **K**, **K4**, **GL**, **T**, **S4**, **Grz**, **B**, **S5***). Then* $L_0 \times^\delta L_1$ *does not have the 'abstract' finite model property.*

Proof. It is straightforward to see that φ_∞ is satisfiable at the root (ω, ω) of $\mathfrak{H}_0 \times^\delta \mathfrak{H}_1$. As concerns the other cases, take the following model $\mathfrak{M}$, either over $\mathfrak{H}_0 \times^\delta \mathfrak{H}_1^{refl}$ or $\mathfrak{H}_0 \times^\delta \mathfrak{H}_1^{univ}$:

$$\mathfrak{M}(v) = \{(n, m) : n \leq \omega,\ m < \omega\}.$$

Then, in both cases, $\big(\mathfrak{M}, (\omega, \omega)\big) \models \varphi^r_\infty$.

Now the theorem follows from Theorems 1 and 2.

4 Discussion

1. Let us emphasise that some results in Theorem 3 are rather surprising, as the corresponding (diagonal-free) products do have the fmp. It is shown in [3, 17] (using filtration) that if L_0 is either **K** or **S5**, and L_1 is axiomatisable by formulas having Horn first-order correspondents (such as e.g., **K**, **K4**, **T**, **S4**, **B**, **S5**), then $L_0 \times L_1$ has the fmp. Moreover, $\mathbf{K} \times \mathbf{K}$ and $\mathbf{K} \times \mathbf{S5}$ even have the fmp w.r.t. product frames [4, 2].
2. Let us next summarise what is known about the fmp of δ-products that are out of the scope of Theorem 3. If *both* components L_0 and L_1 are logics having only transitive frames of arbitrary depth and width (such as, e.g., **K4**, **GL**, **S4**, **Grz**), then it is shown in [5, 6] that already $L_0 \times L_1$ lacks the fmp. So by Prop. 2, $L_0 \times^\delta L_1$ does not have the fmp either. It is not clear, however, whether the 'chessboard-trick' of [6] can be used to extend the proofs of our Theorems 1–3 to cover δ-products with a reflexive (but not necessarily transitive) first component, like e.g. $\mathbf{T} \times^\delta \mathbf{T}$.
3. Though representable diagonal-free cylindric algebras of dimension 2 are the modal algebras of $\mathbf{S5} \times \mathbf{S5}$, two-dimensional *representable cylindric algebras* are not exactly the modal algebras of $\mathbf{S5} \times^\delta \mathbf{S5}$, but those of

$$\mathbf{S5} \times^{\delta'} \mathbf{S5} = \{\psi \in \mathcal{L}_2^\delta : \psi \text{ is valid in } (W, W \times W) \times^\delta (W, W \times W),\ W \text{ is a non-empty set}\}.$$

   (That is, unlike in Prop. 1, only δ-product frames of *rooted* **S5**-frames sharing a common set of worlds are considered in this definition.) It is straightforward to see that $\mathbf{S5} \times^\delta \mathbf{S5} \subseteq \mathbf{S5} \times^{\delta'} \mathbf{S5}$ and that this inclusion is proper: for instance, $\Diamond_0 \delta$ belongs to $\mathbf{S5} \times^{\delta'} \mathbf{S5}$ but not to $\mathbf{S5} \times^\delta \mathbf{S5}$. The propositional modal logic $\mathbf{S5} \times^{\delta'} \mathbf{S5}$ is also connected to two-variable first-order logic with equality, so there are several known proofs showing that $\mathbf{S5} \times^{\delta'} \mathbf{S5}$ is decidable and has the fmp, even w.r.t. 'δ-squares' of universal frames [13, 7]. Perhaps the same is true for $\mathbf{S5} \times^\delta \mathbf{S5}$.
4. As concerns the decision problem for δ-product logics other than $\mathbf{S5} \times^{\delta'} \mathbf{S5}$, not much is known. It is straightforward to extend the proof given in [3] for product logics to δ-products, and show that $L_0 \times^\delta L_1$ is recursively enumerable whenever the class of all frames for each of L_0 and L_1 is recursively first-order definable. However, even having the fmp would not necessarily

help in solving the decision problems. It is shown in [9] that, for many component logics L_0 and L_1 (with **K**, **T**, **K4**, **S4** among them), $L_0 \times^\delta L_1$ is not only not finitely axiomatisable, but it cannot be axiomatised by any set of $\mathcal{L}_2^\delta$-formulas containing finitely many propositional variables. It is not known, however, whether there is some other way of deciding if a finite frame is a frame for such a δ-product logic.

Acknowledgement. Thanks are due to Stanislav Kikot for the problem and interesting discussions.

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