The decision problem of modal product logics with a diagonal, and faulty counter machines

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Abstract

In the propositional modal (and algebraic) treatment of two-variable first-order logic equality is modelled by a 'diagonal' constant, interpreted in square products of universal frames as the identity (also known as the 'diagonal') relation. Here we study the decision problem of products of two *arbitrary* modal logics equipped with such a diagonal. As the presence or absence of equality in two-variable first-order logic does not influence the complexity of its satisfiability problem, one might expect that adding a diagonal to product logics in general is similarly harmless. We show that this is far from being the case, and there can be quite a big jump in complexity, even from decidable to the highly undecidable. Our undecidable logics can also be viewed as new fragments of first-order logic where adding equality changes a decidable fragment to undecidable. We prove our results by a novel application of counter machine problems. While our formalism apparently cannot force reliable counter machine computations directly, the presence of a unique diagonal in the models makes it possible to encode both lossy and insertion-error computations, for the *same* sequence of instructions. We show that, given such a pair of faulty computations, it is then possible to reconstruct a reliable run from them.

1 Introduction

It is well-known that the first-order quantifier $\forall x$ can be considered as an '**S5**-box': a propositional modal $\Box$-operator interpreted over universal frames (that is, relational structures $\langle W, R\rangle$ where $R = W \times W$). The so-called 'standard translation', mapping modal formulas to first-order ones, establishes a validity preserving, bijective connection between the modal logic **S5** and the one-variable fragment of classical first-order logic [42]. The idea of generalising such a propositional approach to full first-order logic was suggested and thoroughly investigated both in modal setting [30, 20, 41], and in algebraic logic [16, 18]. In particular, the bimodal logic $\mathbf{S5} \times \mathbf{S5}$ over two-dimensional (2D) *squares* of universal frames corresponds to the equality and substitution free fragment of two-variable first-order logic, via a translation that maps propositional variables P to binary predicates $\mathsf{P}(x, y)$, the modal boxes $\Box_0$ and $\Box_1$ to the first-order quantifiers $\forall x$ and $\forall y$, and the Boolean connectives to themselves.

In this setting, *equality* between the two first-order variables can be modally 'represented' by extending the bimodal language with a constant δ, interpreted in square frames with universe $W \times W$ as the *diagonal* set

$$\{\langle x, x\rangle : x \in W\}.$$

The resulting modal logic (algebraically, representable 2D cylindric algebras [18]) is now closer to the full two-variable fragment (though $\mathsf{P}(y, x)$-like transposition of variables is still not expressible in it). The generalisation of the modal treatment of full two-variable first-order logic to *products* of two arbitrary modal logics equipped with a diagonal constant (together with modal operators 'simulating' the substitution and transposition of first-order variables) was suggested in [36, 37]. The product construction as a general combination method on modal logics was introduced in [8], and has been extensively studied ever since (see [7, 21] for surveys and references). Two-dimensional product logics can not only be regarded as generalisations of the first-order quantifiers [23], but they are also connected to several other logical formalisms, such as the one-variable fragment of modal and temporal logics, modal and temporal description logics, and spatio-temporal logics. At first sight, the diagonal constant can only be meaningfully used in applications where the domains of the two component frames consist of objects of similar kinds, or at least overlap. However, as modal languages cannot distinguish between isomorphic frames, in fact *any* subset D of a Cartesian product $W_h \times W_v$ can be considered as an interpretation of the diagonal constant, as long as it is both '*horizontally*' and '*vertically*' *unique* in the following sense:

$$\forall x \in W_h,\ \forall y, y' \in W_v\ \big(\langle x, y\rangle, \langle x, y'\rangle \in D\ \rightarrow\ y = y'\big), \tag{1}$$

$$\forall x, x' \in W_h,\ \forall y \in Wv\ \big(\langle x, y\rangle, \langle x', y\rangle \in D\ \rightarrow\ x = x'\big). \tag{2}$$

So, say, in the one-variable constant-domain fragment of first-order temporal (or modal) logics, the diagonal constant can be added in order to single out a set of special 'time-stamped' objects of the domain, provided no special object is chosen twice and at every moment of time (or world along the modal accessibility relation) at most one special object is chosen.

In this paper we study the decision problem of δ-*product logics*: arbitrary 2D product logics equipped with a diagonal. It is well-known that the presence or absence of equality in the two-variable fragment of first-order logic does not influence the coNExpTime-completeness of its validity problem [34, 28, 14]. So one might expect that adding a diagonal to product logics in general is similarly harmless. The more so that decidable product logics like $\mathbf{K} \times \mathbf{K}$ (the bimodal logic of all product frames) remain decidable when one adds modal operators 'simulating' the substitution and transposition of first-order variables [38]. However, we show that adding the diagonal is more dangerous, and there can be quite a big jump in complexity. In some cases, the global consequence relation of product logics can be reduced the validity-problem of the corresponding δ-products (Prop. 2). We also show (Theorems 2, 4) that if L is any logic having an infinite rooted frame where each point can be accessed by at most one step from the root, then both $\mathbf{K} \times^{\delta} L$ and $\mathbf{K4.3} \times^{\delta} L$ are undecidable (here $\mathbf{K}$ is the unimodal logic of all frames, and $\mathbf{K4.3}$ is the unimodal logic of linear orders). Some notable consequences of these results are:

(i) $\mathbf{K} \times^{\delta} \mathbf{S5}$ is undecidable, (while $\mathbf{K} \times \mathbf{S5}$ is coNExpTime-complete [24], and even the global consequence relation of $\mathbf{K} \times \mathbf{S5}$ is decidable in co2NExpTime [43, 33]).

(ii) $\mathbf{K4.3} \times^{\delta} \mathbf{S5}$ is undecidable (while $\mathbf{K4.3} \times \mathbf{S5}$ is decidable in 2ExpTime [31]).

(iii) $\mathbf{K}\times^{\delta}\mathbf{K}$ is undecidable (while $\mathbf{K}\times\mathbf{K}$ is decidable [8], though not in ElementaryTime [13]).

See also Table 1 for some known results on product logics, and how our present results on δ-products compare with them.

While all the above δ-product logics are recursively enumerable (Theorem 1), we also show that in some cases decidable product logics can turn highly undecidable by adding a diagonal. For instance, both $\mathbf{K}\times^{\delta}\mathbf{S5}$ and $\mathbf{K}\times^{\delta}\mathbf{K}$ when restricted to finite (but unbounded) product frames result in non-recursively enumerable logics (Theorem 3). Also, $\mathsf{Logic\_of}\langle\omega,<\rangle\times^{\delta}\mathbf{S5}$ is Π^1_1-hard (Theorem 5). On the other hand, the unbounded width of the second-component frames seems to be essential in obtaining these results. Adding a diagonal to decidable product logics of the form $\mathbf{K}\times\mathbf{Alt}(n)$, $\mathbf{S5}\times\mathbf{Alt}(n)$, and $\mathbf{Alt}(m)\times\mathbf{Alt}(n)$ results in decidable logics, sometimes even with the same upper bounds that are known for the products (Theorems 6 and 7) (here $\mathbf{Alt}(n)$ is the unimodal logic of frames where each point has at most n successors for some $0<n<\omega$).

Our undecidable δ-product logics can also be viewed as new fragments of first-order logic where adding equality changes a decidable fragment to undecidable. (A well-known such fragment is the Gödel class [11, 12].) In particular, consider the following '2D extension' of the standard translation [9], from bimodal formulas to three-variable first-order formulas having two free variables x and y and a built-in binary predicate R:

$$\begin{aligned}
\mathsf{P}^{\dagger} &:= \mathsf{P}(x,y), \quad \text{for propositional variables } \mathsf{P},\\
(\neg\phi) &:= \neg\phi^{\dagger} \quad \text{and} \quad (\phi\wedge\psi)^{\dagger} := \phi^{\dagger}\wedge\psi^{\dagger},\\
(\Box_0\phi)^{\dagger} &:= \forall z\,\big(\mathsf{R}(x,z)\to\phi^{\dagger}(z/x,y)\big),\\
(\Box_1\phi)^{\dagger} &:= \forall z\,\big(\mathsf{R}(y,z)\to\phi^{\dagger}(x,z/y)\big).
\end{aligned}$$

It is straightforward to see that, for any bimodal formula ϕ, ϕ is satisfiable in the (decidable) modal product logic $\mathbf{K}\times\mathbf{K}$ iff $\phi^{\dagger}$ is satisfiable in first-order logic. So the image of † is a decidable fragment of first-order logic that becomes undecidable when equality is added.

Our results show that in many cases the presence of a *single* proposition (the diagonal) with the 'horizontal' and 'vertical' uniqueness properties (1)–(2) is enough to cause undecidability of 2D product logics. If each of the component logics has a *difference operator*, then their product can express 'horizontal' and 'vertical' uniqueness of *any* proposition. For example, this is the case when each component is either the unimodal logic **Diff** of all frames of the form $\langle W,\neq\rangle$, or a logic determined by strict linear orders such as **K4.3** or $\mathsf{Logic\_of}\langle\omega,<\rangle$. So our Theorems 4 and 5 can be regarded as generalisations of the undecidability results of [32] on 'linear'$\times$'linear'-type products, and those of [17] on 'linear'$\times$**Diff**-type products.

On the proof methods. Even if 2D product structures are always grid-like by definition, there are two issues one needs to deal with in order to encode grid-based complex problems into them:

(i) to generate infinity, even when some component structure is not transitive, and

(ii) somehow to 'access' or 'refer to' neighbouring-grid points, even when there is no 'next-time' operator in the language, and/or the component structures are transitive or even universal.

	validity of product logic	global consequence of product logic	**validity of δ-product logic**
S5 $\times$ **S5**	coNExpTime-complete [34, 28, 14, 24]	same as validity	coNExpTime-complete [34, 28, 14]
K $\times$ **S5**	coNExpTime-complete [24]	decidable in co2NExpTime [43, 33]	**undecidable Cor. 2**
K $\times$ **K**	decidable [8] not in ElementaryTime [13]	undecidable [24]	**undecidable Cor. 1**
K4.3 $\times$ **S5**	decidable in 2ExpTime [31] coNExpTime-hard [24]	same as validity	**undecidable Cor. 3**
K4 $\times$ **S5**	decidable in coN2ExpTime [8] coNExpTime-hard [24]	same as validity	**?**
K4 $\times$ **K**	decidable [43] not in ElementaryTime [13]	undecidable [15]	**undecidable Cor. 1**
K4 $\times$ **K4**	undecidable [10]	same as validity	**undecidable Prop. 1**
K $\times$ **Alt**(n)	decidable in coNExpTime ($n > 1$) in ExpTime ($n = 1$) [7]	undecidable	**decidable in coNExpTime Thm. 6**

Table 1: Product vs. δ-product logics.

When both component structures are transitive, then (i) is not a problem. If in addition component structures of arbitrarily large depths are available, then (ii) is usually solved by 'diagonally' encoding the $\omega \times \omega$-grid, and then use reductions of tiling or Turing machine problems [25, 32, 10]. When both components can express the uniqueness of any proposition (like strict linear orders or the difference operator), then it is also possible to make direct use of the grid-like nature of product structures and obtain undecidability by forcing reliable counter machine computations [17]. However, δ-product logics of the form $L \times^{\delta} \mathbf{S5}$ apparently neither can force such computations directly, nor they can diagonally encode the $\omega \times \omega$-grid. Instead, we prove our lower bound results by a novel application of counter machine problems. The presence of a unique diagonal in the models makes it possible to encode both *lossy* and *insertion-error* computations, for the *same* sequence of instructions. We then show (Prop. 3) that, given such a pair of faulty computations, one can actually reconstruct a reliable run from them. The upper bound results are shown by a straightforward selective filtration.

The structure of the paper is as follows. Section 2 provides all the necessary definitions. In Section 3 we establish connections between our logics and other formalisms, and discuss some consequences of these connections on the decision problem of δ-products. In Section 4 we introduce counter machines, and discuss how reliable counter machine computations can be approximated by faulty (lossy and insertion-error) ones. Then in Sections 5 and 6 we state and prove our undecidability results on δ-products having a $\mathbf{K}$ or a 'linear' component, respectively. The decidability results are proved in Section 7. Finally, in Section 8 we discuss some related open problems.

2 δ-product logics

In what follows we assume that the reader is familiar with the basic notions in modal logic and its possible world semantics (see [3, 5] for reference). Below we summarise the necessary notions and notation for our 3-modal case only, but we will use them throughout for the uni- and bimodal cases as well. We define our *formulas* by the following grammar:

$$\phi := \ \mathsf{P} \mid \delta \mid \neg\phi \mid \phi \wedge \psi \mid \Box_h \phi \mid \Box_v \phi,$$

where P ranges over an infinite set of propositional variables. We use the usual abbreviations $\vee$, $\rightarrow$, $\leftrightarrow$, $\bot := \mathsf{P} \wedge \neg\mathsf{P}$, $\Diamond_i := \neg\Box_i\neg$, and also

$$\Diamond_i^+ \phi := \ \phi \vee \Diamond_i \phi, \qquad\qquad \Box_i^+ \phi := \ \phi \wedge \Box_i \phi,$$

for $i = h, v$. (The subscripts are indicative of the 2D intuition: h for 'horizontal' and v for 'vertical'.)

A δ-*frame* is a tuple $\mathfrak{F} = \langle W, R_h, R_v, D\rangle$ where R_i are binary relations on the non-empty set W, and D is a subset of W. We call $\mathfrak{F}$ *rooted* if there is some w such that wR^*v for all $v \in W$, for the reflexive and transitive closure R^* of $R := R_h \cup R_v$. A *model based on* $\mathfrak{F}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, \nu\rangle$, where ν is a function mapping propositional variables to subsets of W. The *truth relation* $\mathfrak{M}, w \models \phi$ is defined, for all $w \in W$, by induction on ϕ as usual. In particular,

$$\mathfrak{M}, w \models \delta \quad \text{iff} \quad w \in D.$$

We say that ϕ is *satisfied in* $\mathfrak{M}$, if there is $w \in W$ with $\mathfrak{M}, w \models \phi$. We write $\mathfrak{M} \models \phi$, if $\mathfrak{M}, w \models \phi$ for every $w \in W$. Given a set L of formulas, we write $\mathfrak{M} \models L$ if $\mathfrak{M} \models \phi$ for every

ϕ in L. Given formulas ϕ and ψ, we write $\phi \models^*_L \psi$ iff $\mathfrak{M} \models \psi$ for every model $\mathfrak{M}$ such that $\mathfrak{M} \models L \cup \{\phi\}$.

We say that ϕ is *valid in* $\mathfrak{F}$, if $\mathfrak{M} \models \phi$ for every model $\mathfrak{M}$ based on $\mathfrak{F}$. If every formula in a set L is valid in $\mathfrak{F}$, then we say that $\mathfrak{F}$ is a *frame for* L. We let $\mathsf{Fr}\, L$ denote the class of all frames for L. For any class $\mathcal{C}$ of δ-frames, we let

$$\mathsf{Logic\_of}\, \mathcal{C} := \{\phi : \phi \text{ is a formula valid in every member of } \mathcal{C}\}.$$

We call a set L of formulas a *Kripke complete logic* if $L = \mathsf{Logic\_of}\, \mathcal{C}$ for some class $\mathcal{C}$. A Kripke complete logic L such that for all formulas ϕ and ψ, $\phi \models^*_L \psi$ iff $\mathfrak{M} \models \phi$ implies $\mathfrak{M} \models \psi$ for every model $\mathfrak{M}$ based on a frame for L, is called *globally Kripke complete*.

We are interested in some special 'two-dimensional' δ-frames. Given unimodal Kripke frames $\mathfrak{F}_h = \langle W_h, R_h \rangle$ and $\mathfrak{F}_v = \langle W_v, R_v \rangle$, their *product* is the bimodal frame

$$\mathfrak{F}_h \times \mathfrak{F}_v := \langle W_h \times W_v, \overline{R}_h, \overline{R}_v \rangle,$$

where $W_h \times W_v$ is the Cartesian product of sets W_h and W_v and the binary relations $\overline{R}_h$ and $\overline{R}_v$ are defined by taking, for all $x, x' \in W_h$, $y, y' \in W_v$,

$$\langle x, y \rangle \overline{R}_h \langle x', y' \rangle \quad \text{iff} \quad x R_h x' \text{ and } y = y',$$
$$\langle x, y \rangle \overline{R}_v \langle x', y' \rangle \quad \text{iff} \quad y R_v y' \text{ and } x = x'.$$

The *δ-product* of $\mathfrak{F}_h$ and $\mathfrak{F}_v$ is the δ-frame

$$\mathfrak{F}_h \times^\delta \mathfrak{F}_v := \langle W_h \times W_v, \overline{R}_h, \overline{R}_v, \mathsf{id} \rangle,$$

where $\langle W_h \times W_v, \overline{R}_h, \overline{R}_v \rangle = \mathfrak{F}_h \times \mathfrak{F}_v$ and

$$\mathsf{id} = \{\langle x, x \rangle : x \in W_h \cap W_v\}.$$

For classes $\mathcal{C}_h$ and $\mathcal{C}_v$ of unimodal frames, we define

$$\mathcal{C}_h \times^\delta \mathcal{C}_v = \{\mathfrak{F}_h \times^\delta \mathfrak{F}_v : \mathfrak{F}_i \in \mathcal{C}_i, \text{ for } i = h, v\}.$$

Now, for $i = h, v$, let L_i be a Kripke complete unimodal logic in the language with $\Diamond_i$. The *δ-product* of L_h and L_v is defined as

$$L_h \times^\delta L_v := \mathsf{Logic\_of}\, (\mathsf{Fr}\, L_h \times^\delta \mathsf{Fr}\, L_v).$$

As a generalisation of the modal approximation of two-variable first-order logic, it might be more 'faithful' to consider

$$L_h \times^\delta_{sq} L_v := \{\phi : \phi \text{ is valid in } \mathfrak{F}_h \times^\delta \mathfrak{F}_v, \text{ for some rooted } \mathfrak{F}_i = \langle W_i, R_i \rangle \text{ in } \mathsf{Fr}\, L_i,\ i = h, v, \text{ such that } W_h = W_v\},$$

or, in case $L_h = L_v = L$, even

$$L \times^\delta_{sqf} L := \{\phi : \phi \text{ is valid in } \mathfrak{F} \times^\delta \mathfrak{F}, \text{ for some rooted } \mathfrak{F} \in \mathsf{Fr}\, L\}.$$

Then $\mathbf{S5} \times^\delta_{sq} \mathbf{S5} = \mathbf{S5} \times^\delta_{sqf} \mathbf{S5}$ indeed corresponds to the transposition-free fragment of two-variable first-order logic. However, $\mathbf{S5} \times^\delta \mathbf{S5}$ is properly contained in $\mathbf{S5} \times^\delta_{sq} \mathbf{S5}$: for instance

$\Diamond_h\delta$ belongs to the latter but not to the former. In general, clearly we always have $L_h \times^\delta L_v \subseteq L_h \times^\delta_{sq} L_v$ and $L \times^\delta_{sq} L \subseteq L \times^\delta_{sqf} L$, whenever $L_h = L_v = L$. Also, it is not hard to give examples when the three definitions result in three different logics. Throughout, we formulate all our results for the $L_h \times^\delta L_v$ cases only, but each and every of them holds for the corresponding $L_h \times^\delta_{sq} L_v$ as well (and also for $L \times^\delta_{sqf} L$ when it is meaningful to consider the same L as both components).

Given a set L of formulas, we are interested in the following decision problems:

L-VALIDITY: Given a formula ϕ, does it belong to L?

If this problem is (un)decidable, we simply say that 'L is (un)decidable'. L-validity is the 'dual' of

L-SATISFIABILITY: Given a formula ϕ, is there a model $\mathfrak{M}$ such that $\mathfrak{M} \models L$ and ϕ is satisfied in $\mathfrak{M}$?

Clearly, if $L = \mathsf{Logic\_of}\,\mathcal{C}$ then L-satisfiability is the same as

$\mathcal{C}$-SATISFIABILITY: Given a formula ϕ, is there a frame $\mathfrak{F} \in \mathcal{C}$ such that ϕ is satisfied in a model based on $\mathfrak{F}$?

We also consider

GLOBAL L-CONSEQUENCE: Given formulas ϕ and ψ, does $\phi \models^*_L \psi$ hold?

Notation. Our notation is mostly standard. In particular, we denote by R^+ the *reflexive closure* of a binary relation R. The cardinality of a set X is denoted by $|X|$. For each natural number $k < \omega$, we also consider k as the finite ordinal $k = \{0, \dots, k-1\}$.

3 Decidability of δ-products: what to expect?

To begin with, the following proposition is straightforward from the definitions:

Proposition 1. *$L_h \times^\delta L_v$ is always a conservative extension of $L_h \times L_v$.*

So it follows from the undecidability results of [10] on the corresponding product logics that $L_h \times^\delta L_v$ is undecidable, whenever both L_h and L_v have only *transitive* frames and have frames of *arbitrarily large depths*. For example, $\mathbf{K4} \times^\delta \mathbf{K4}$ is undecidable, where $\mathbf{K4}$ is the unimodal logic of all transitive frames.

Next, we establish connections between the global consequence relation of some product logics and the corresponding δ-products. To begin with, we introduce an operation on frames that we call *disjoint union with a spy-point*. Given unimodal frames $\mathfrak{F}_i = \langle W_i, R_i \rangle$, $i \in I$, for some index set I, and a fresh point r, we let

$$\bigcup_{i\in I}^{r} \mathfrak{F}_i := \langle W, R \rangle,$$

where

$$W = \{r\} \cup \{\langle w, i\rangle : i \in I,\ w \in W_i\}, \quad \text{and}$$
$$R = \big\{\langle r, \langle w, i\rangle\rangle : w \in W_i,\ i \in I\big\} \cup \big\{\langle\langle w, i\rangle, \langle w', i\rangle\rangle : w, w' \in W_i,\ wR_iw',\ i \in I\big\}.$$

Note that the spy-point technique is well-known in hybrid logic [4].

Proposition 2. *If L_h and L_v are Kripke complete logics such that both $\mathsf{Fr}\,L_h$ and $\mathsf{Fr}\,L_v$ are closed under the 'disjoint union with a spy-point' operation and $L_h \times L_v$ is globally Kripke complete, then the global $L_h \times L_v$-consequence is reducible to $L_h \times^\delta L_v$-validity.*

Proof. We show that for all bimodal (δ-free) formulas ϕ, ψ,

$$\phi \models^*_{L_h \times L_v} \psi \qquad \text{iff} \qquad \big((\mathsf{univ}^\delta \wedge \Box_h \Box_v \phi) \to \Box_h \Box_v \psi\big) \in L_h \times^\delta L_v,$$

where

$$\mathsf{univ}^\delta := \Box_h \Diamond_v \delta \wedge \Box_h \Box_h \Diamond_v \delta \wedge \Box_v \Diamond_h \delta \wedge \Box_v \Box_v \Diamond_h \delta.$$

$\Rightarrow$: Suppose that $\mathfrak{M}, \langle r_h, r_v \rangle \models \mathsf{univ}^\delta \wedge \Box_h \Box_v \phi \wedge \Diamond_h \Diamond_v \neg\psi$ in a model $\mathfrak{M}$ that is based on $\mathfrak{F}_h \times^\delta \mathfrak{F}_v$, for some frames $\mathfrak{F}_i = \langle W_i, R_i \rangle$ in $\mathsf{Fr}\,L_i$, $i = h, v$. Then there exist x_h, x_v such that $r_h R_h x_h$, $r_v R_v x_v$ and $\mathfrak{M}, \langle x_h, x_v \rangle \models \neg\psi$. For $i = h, v$, let $\mathfrak{G}_i$ be the subframe of $\mathfrak{F}_i$ generated by point x_i, and let $\mathfrak{N}$ be the restriction of $\mathfrak{M}$ to $\mathfrak{G}_h \times \mathfrak{G}_v$. Then

$$\mathfrak{N} \models L_h \times L_v \quad \text{and} \quad \mathfrak{N}, \langle x_h, x_v \rangle \models \neg\psi. \tag{3}$$

We claim that

$$r_i R_i w, \text{ for all } w \text{ in } \mathfrak{G}_i \text{ and } i = h, v. \tag{4}$$

Indeed, let $i = h$. We prove (4) by induction on the smallest number n of R_h-steps needed to access w from x_h. If $n = 0$ then we have $r_h R_h x_h$. Now suppose inductively that (4) holds for all w in $\mathfrak{G}_h$ that are accessible in $\leq n$ R_h-steps from x_h for some $n < \omega$, and let w' be accessible in $n + 1$ R_h-steps. Then there is w in $\mathfrak{G}_h$ that is accessible in n steps and $w R_h w'$. Thus $r_h R_h w$ by the IH, and so $\mathfrak{M}, \langle w', r_v \rangle \models \Diamond_v \delta$ by univ^δ. Therefore, we have $w' \in W_v$ and $r_v R_v w'$. Then $\mathfrak{M}, \langle r_h, w' \rangle \models \Diamond_h \delta$ again by univ^δ, and so $r_h R_h w'$ as required. The $i = v$ case is similar.

Now it follows from $\mathfrak{M}, \langle r_h, r_v \rangle \models \Box_h \Box_v \phi$ and (4) that $\mathfrak{N} \models \phi$. Therefore, $\phi \not\models^*_{L_h \times L_v} \psi$ by (3).

$\Leftarrow$: Suppose that $\mathfrak{M} \models \phi$ and $\mathfrak{M}, w \models \neg\psi$ in some model $\mathfrak{M}$ with $\mathfrak{M} \models L_h \times L_v$. As $L_h \times L_v$ is globally Kripke complete, we may assume that $\mathfrak{M} = \langle \mathfrak{F}_h \times \mathfrak{F}_v, \mu \rangle$ for some frames $\mathfrak{F}_i = \langle W_i, R_i \rangle$ in $\mathsf{Fr}\,L_i$, $i = i, h$. Let $\mathfrak{F}_h^\alpha$, $\alpha < |W_v|$, be $|W_v|$-many copies of $\mathfrak{F}_h$, and $\mathfrak{F}_v^\beta$, $\beta < |W_h|$, be $|W_h|$-many copies of $\mathfrak{F}_v$. Take some fresh point r and define

$$\mathfrak{G}_h = \langle U_h, S_h \rangle := \bigcup_{\alpha < |W_v|}^{r} \mathfrak{F}_h^\alpha \qquad \text{and} \qquad \mathfrak{G}_v = \langle U_v, S_v \rangle := \bigcup_{\beta < |W_h|}^{r} \mathfrak{F}_v^\beta.$$

Then by our assumption, $\mathfrak{G}_i$ is a frame for L_i, for $i = h, v$. Define a model $\mathfrak{N} := \langle \mathfrak{G}_h \times^\delta \mathfrak{G}_v, \nu \rangle$ by taking, for all propositional variables P,

$$\nu(\mathsf{P}) := \big\{ \big\langle \langle x, \alpha \rangle, \langle y, \beta \rangle \big\rangle : \langle x, y \rangle \in \mu(\mathsf{P}) \big\}.$$

Then $\mathfrak{N}, \langle r, r \rangle \models \Box_h \Box_v \phi \wedge \Diamond_h \Diamond_v \neg\psi$. As $|U_h| = |U_v|$ and $\mathsf{Fr}\,L_i$ is closed under isomorphic copies for $i = h, v$, we can actually assume that $U_h = U_v$, and so $\mathfrak{N}, \langle r, r \rangle \models \mathsf{univ}^\delta$. $\Box$

Corollary 1. $\mathbf{K} \times^\delta \mathbf{K}$ *and* $\mathbf{K} \times^\delta \mathbf{K4}$ *are both undecidable.*

Proof. It is not hard to check that the 2D product logics $\mathbf{K} \times \mathbf{K}$ and $\mathbf{K} \times \mathbf{K4}$ satisfy the requirements in Prop. 2 (cf. [7, Thm.5.12] for global Kripke completeness). A reduction of, say, the $\omega \times \omega$-tiling problem [2] shows that global $\mathbf{K} \times \mathbf{K}$-consequence is undecidable [24], and so the undecidability of $\mathbf{K} \times^{\delta} \mathbf{K}$ follows by Prop. 2. It is shown in [15] that the reduction of $\mathbf{K4}$ to global $\mathbf{K}$-consequence [40] can be 'lifted' to the product level, and so $\mathbf{K4} \times \mathbf{K4}$ is reducible to global $\mathbf{K} \times \mathbf{K4}$-consequence. Therefore, the latter is undecidable [10], and so the undecidability of $\mathbf{K} \times^{\delta} \mathbf{K4}$ follows by Prop. 2. $\square$

Note that we can also make Prop. 2 work for logics having only *reflexive* frames by making the 'spy-point' reflexive, and using a slightly different 'translation':

$$\phi \models^{*}_{L_h \times L_v} \psi \qquad \text{iff} \qquad \big((\mathsf{univ}^{\delta} \wedge \Box_h \mathsf{P} \wedge \Box_v \mathsf{P} \wedge \Box_h \Box_v (\neg \mathsf{P} \to \phi) \to \Box_h \Box_v (\neg \mathsf{P} \to \psi)\big) \in L_h \times^{\delta} L_v,$$

where P is a fresh propositional variable.

However, logics having only symmetric frames (like $\mathbf{S5}$), or having only frames with bounded width (like $\mathbf{K4.3}$ or $\mathbf{Alt}(n)$) are not closed under the 'disjoint union with a spy-point' operation, and so Prop. 2 does not apply to their products. It turns out that in some of these cases such a reduction is either not useful in establishing undecidability of δ-products, or does not even exist. While global $\mathbf{K} \times \mathbf{S5}$-consequence is reducible to $\mathbf{PDL} \times \mathbf{S5}$-validity[1], and so decidable in co2NExpTime [43, 33], $\mathbf{K} \times^{\delta} \mathbf{S5}$ is shown to be undecidable in Theorem 2 below. While $\mathbf{K} \times^{\delta} \mathbf{Alt}(n)$ is decidable by Theorem 6 below, the undecidability of global $\mathbf{K} \times \mathbf{Alt}(n)$-consequence can again be shown by a straightforward reduction of the $\omega \times \omega$-tiling problem.

Finally, the following general result is a straightforward generalisation of the similar theorem of [8] on product logics. It is an easy consequence of the recursive enumerability of the consequence relation of (many-sorted) first-order logic:

Theorem 1. *If L_h and L_v are Kripke complete logics such that both $\mathsf{Fr}\, L_h$ and $\mathsf{Fr}\, L_v$ are recursively first-order definable in the language having a binary predicate symbol, then $L_h \times^{\delta} L_v$ is recursively enumerable.*

4 Reliable counter machines and faulty approximations

A *Minsky* [27] or *counter machine* M is described by a finite set Q of states, an initial state $q_{\mathsf{ini}} \in Q$, a set $H \subseteq Q$ of terminal states, a finite set $C = \{c_0, \dots, c_{N-1}\}$ of counters with $N > 1$, a finite nonempty set $I_q \subseteq Op_C \times Q$ of instructions, for each $q \in Q - H$, where each operation in Op_C is one of the following forms, for some $i < N$:

- c_i^{++} (*increment counter c_i by one*),
- c_i^{--} (*decrement counter c_i by one*),
- $c_i^{??}$ (*test whether counter c_i is empty*).

[1]Here $\mathbf{PDL}$ denotes Propositional Dynamic Logic.

For each $\alpha \in Op_C$, we will consider three different kinds of semantics: *reliable* (as described above), *lossy* [26] (when counters can spontaneously decrease, both before and after performing α), and *insertion-error* [29] (when counters can spontaneously increase, both before and after performing α).

A *configuration* of M is a tuple $\langle q, \vec{c}\rangle$ with $q \in Q$ representing the current state, and an N-tuple $\vec{c} = \langle c_0, \dots, c_{N-1}\rangle$ of natural numbers representing the current contents of the counters. Given $\alpha \in Op_C$, we say that *there is a reliable α-step* between configurations $\langle q, \vec{c}\rangle$ and $\langle q', \vec{c}\,'\rangle$ (written $\langle q, \vec{c}\rangle \rightarrow^{\alpha} \langle q', \vec{c}\,'\rangle$) iff $\langle \alpha, q'\rangle \in I_q$ and

- if $\alpha = c_i^{++}$ then $c_i' = c_i + 1$ and $c_j' = c_j$ for $j \neq i$, $j < N$;
- if $\alpha = c_i^{--}$ then $c_i' = c_i - 1$ and $c_j' = c_j$ for $j \neq i$, $j < N$;
- if $\alpha = c_i^{??}$ then $c_i' = c_i = 0$ and $c_j' = c_j$ for $j < N$.

We say that *there is a lossy α-step* between configurations $\langle q, \vec{c}\rangle$ and $\langle q', \vec{c}\,'\rangle$ (and we write $\langle q, \vec{c}\rangle \rightarrow^{\alpha}_{lossy} \langle q', \vec{c}\,'\rangle$) iff $\langle \alpha, q'\rangle \in I_q$ and

- if $\alpha = c_i^{++}$ then $c_i' \leq c_i + 1$ and $c_j' \leq c_j$ for $j \neq i$, $j < N$;
- if $\alpha = c_i^{--}$ then $c_i' \leq c_i - 1$ and $c_j' \leq c_j$ for $j \neq i$, $j < N$;
- if $\alpha = c_i^{??}$ then $c_i' = 0$ and $c_j' \leq c_j$ for $j < N$.

Finally, we say that *there is an insertion-error α-step* between configurations $\langle q, \vec{c}\rangle$ and $\langle q', \vec{c}\,'\rangle$ (written $\langle q, \vec{c}\rangle \rightarrow^{\alpha}_{i\_err} \langle q', \vec{c}\,'\rangle$) iff $\langle \alpha, q'\rangle \in I_q$ and

- if $\alpha = c_i^{++}$ then $c_i' \geq c_i + 1$ and $c_j' \geq c_j$ for $j \neq i$, $j < N$;
- if $\alpha = c_i^{--}$ then $c_i' \geq c_i - 1$ and $c_j' \geq c_j$ for $j \neq i$, $j < N$;
- if $\alpha = c_i^{??}$ then $c_i = 0$ and $c_j' \geq c_j$ for $j < N$.

Now suppose that a sequence $\vec{\tau} = \big\langle \langle \alpha_n, q_n\rangle : 0 < n < B \big\rangle$ of instructions of M is given for some $0 < B \leq \omega$. We say that a sequence $\vec{\varrho} = \big\langle \langle q_n, \vec{c}(n)\rangle : n < B \big\rangle$ of configurations is a *reliable $\vec{\tau}$-run of M* if

(i) $q_0 = q_{\mathsf{ini}}$, $\vec{c}(0) = \vec{0}$, and

(ii) $\langle q_{n-1}, \vec{c}(n-1)\rangle \rightarrow^{\alpha_n} \langle q_n, \vec{c}(n)\rangle$ holds for every $0 < n < B$.

A *reliable run* is a reliable $\vec{\tau}$-run for some $\vec{\tau}$. Similarly, a sequence $\vec{\varrho}$ satisfying (i) is called a *lossy $\vec{\tau}$-run* if we have $\langle q_{n-1}, \vec{c}(n-1)\rangle \rightarrow^{\alpha_n}_{lossy} \langle q_n, \vec{c}(n)\rangle$, and an *insertion-error $\vec{\tau}$-run* if we have $\langle q_{n-1}, \vec{c}(n-1)\rangle \rightarrow^{\alpha_n}_{i\_err} \langle q_n, \vec{c}(n)\rangle$, for every $0 < n < B$. (Note that in order to simplify the presentation, in each case we only consider runs that start at state q_{ini} with all-zero counters.)

Observe that, for any given $\vec{\tau}$, if there exists a reliable $\vec{\tau}$-run, then it is unique. The following statement says that this unique reliable $\vec{\tau}$-run can be 'approximated' by a $\langle$lossy, insertion-error$\rangle$-pair of $\vec{\tau}$-runs:

Proposition 3. (faulty approximation)
Given any sequence $\vec{\tau}$ of instructions, there exists a reliable $\vec{\tau}$-run iff there exist both lossy and insertion-error $\vec{\tau}$-runs.

Proof. The $\Rightarrow$ direction is obvious, as each reliable $\vec{\tau}$-run is both a lossy and an insertion-error $\vec{\tau}$-run as well. For the $\Leftarrow$ direction, suppose that $\vec{\tau} = \big\langle \langle \alpha_n, q_n \rangle : 0 < n < B \big\rangle$ for some $B \leq \omega$, $\big\langle \langle q_n, \vec{c}^{\,\circ}(n) \rangle : n < B \big\rangle$ is a lossy $\vec{\tau}$-run, and $\big\langle \langle q_n, \vec{c}^{\,\bullet}(n) \rangle : n < B \big\rangle$ is an insertion-error $\vec{\tau}$-run. We claim that there is a sequence $\langle \vec{c}(n) : n < B \rangle$ of N-tuples of natural numbers such that, for every $n < B$,

(a) $c_i^\circ(n) \leq c_i(n) \leq c_i^\bullet(n)$ for every $i < N$,

(b) if $n > 0$ then $\langle q_{n-1}, \vec{c}(n-1) \rangle \to^{\alpha_n} \langle q_n, \vec{c}(n) \rangle$.

It would follow that $\big\langle \langle q_n, \vec{c}(n) \rangle : n < B \big\rangle$ is a reliable $\vec{\tau}$-run as required.

We prove the claim by induction on n. To begin with, we let $\vec{c}(0) := \vec{0}$. Now suppose that (a) and (b) hold for all $k < n$ for some n with $0 < n < B$. For each $i < N$, we let

$$c_i(n) := \begin{cases} c_i(n-1) + 1, & \text{if } \alpha_n = c_i^{++}, \\ c_i(n-1) - 1, & \text{if } \alpha_n = c_i^{--}, \\ c_i(n-1), & \text{if } \alpha_n = c_i^{??} \text{ or } \alpha_n \in \{c_j^{++}, c_j^{--}, c_j^{??}\} \text{ for } j \neq i. \end{cases}$$

We need to check that (a) and (b) hold for n. There are several cases, depending on α_n. If $\alpha_n = c_i^{??}$ then, by $\langle q_{n-1}, \vec{c}^{\,\circ}(n-1) \rangle \to^{\alpha_n}_{lossy} \langle q_n, \vec{c}^{\,\circ}(n) \rangle$, the IH(a), and $\langle q_{n-1}, \vec{c}^{\,\bullet}(n-1) \rangle \to^{\alpha_n}_{i\_err} \langle q_n, \vec{c}^{\,\bullet}(n) \rangle$, we have

$$c_j^\circ(n) \leq c_j^\circ(n-1) \leq c_j(n-1) = c_j(n) \leq c_j^\bullet(n-1) \leq c_j^\bullet(n) \quad \text{for all } j \neq i.$$

Also, $c_i^\bullet(n-1) = 0$ by $\langle q_{n-1}, \vec{c}^{\,\bullet}(n-1) \rangle \to^{\alpha_n}_{i\_err} \langle q_n, \vec{c}^{\,\bullet}(n) \rangle$. So by the IH(a), we have $c_i(n-1) = 0$, and so $c_i(n) = 0$ and $\langle q_{n-1}, \vec{c}(n-1) \rangle \to^{\alpha_n} \langle q_n, \vec{c}(n) \rangle$. As $\langle q_{n-1}, \vec{c}^{\,\circ}(n-1) \rangle \to^{\alpha_n}_{lossy} \langle q_n, \vec{c}^{\,\circ}(n) \rangle$, we have $c_i^\circ(n) = 0$. Thus $c_i^\circ(n) = c_i(n) = c_i^\bullet(n-1) = 0 \leq c_i^\bullet(n)$, as required. The other cases are straightforward and left to the reader. □

In each of our lower bound proofs we will use 'faulty approximation', together with one of the following problems on reliable counter machine runs:

<u>CM non-termination:</u> (Π_1^0-hard [27])

Given a counter machine $\mathcal{M}$, does $\mathcal{M}$ have an infinite reliable run?

<u>CM reachability:</u> (Σ_1^0-hard [27])

Given a counter machine $\mathcal{M}$, and a state q_{fin}, does $\mathcal{M}$ have a reliable run reaching q_{fin}?

<u>CM recurrence:</u> (Σ_1^1-hard [1])

Given a counter machine $\mathcal{M}$ and a state q_r, does $\mathcal{M}$ have a reliable run that visits q_r infinitely often?

5 Undecidable δ-products with a K-component

For each $0 < k \leq \omega$, we call any frame $\langle k, R \rangle$ a *k-fan* if

$$\{\langle 0, n \rangle : 0 < n < k\} \subseteq R. \tag{5}$$

Theorem 2. *Let L be any Kripke complete logic having an ω-fan among its frames. Then* $\mathbf{K} \times^\delta L$ *is undecidable.*

Corollary 2. $\mathbf{K} \times^{\delta} \mathbf{S5}$ *is undecidable.*

We prove Theorem 2 by reducing the 'CM non-termination' problem to $L_h \times^{\delta} L_v$-satisfiability. Let $\mathfrak{M}$ be a model based on the δ-product of some frame $\mathfrak{F}_h = \langle W_h, R_h \rangle$ in $\mathsf{Fr}\, L_h$ and some frame $\mathfrak{F}_v = \langle W_v, R_v \rangle$ in $\mathsf{Fr}\, L_v$. First, we generate an $\omega \times \omega$-grid in $\mathfrak{M}$. Let grid be the conjunction of the formulas

$$\Box_v^+ \Diamond_h \delta, \tag{6}$$

$$\Box_h \Diamond_v (\Diamond_h \delta \wedge \Box_h \delta). \tag{7}$$

Claim 2.1. (grid generation)
If $\mathfrak{M}, \langle r_h, r_v \rangle \models \mathsf{grid}$ then there exist points $\langle x_n \in W_h \cap W_v : n < \omega \rangle$ such that, for all $n < \omega$,

(i) *$r_h R_h x_n$,*

(ii) *$x_0 = r_v$, and if $n > 0$ then $x_0 R_v x_n$,*

(iii) *if $n > 0$ then $x_{n-1} R_h x_n$,*

(iv) *if $n > 0$ then x_n is the only R_h-successor of x_{n-1}.*

(We do not claim that all the x_n are distinct.)

Proof. By induction on n. Let $x_0 := r_v$. Then (i) holds by (6). Now suppose inductively that we have $\langle x_k : k < n \rangle$ satisfying (i)–(iv) for some $0 < n < \omega$. Then by (7), there is $x_n \in W_v$ such that $x_0 R_v x_n$ and $\mathfrak{M}, \langle x_{n-1}, x_n \rangle \models \Diamond_h \delta \wedge \Box_h \delta$. Therefore, $x_n \in W_h$, $x_{n-1} R_h x_n$, and x_n is the only R_h-successor of x_{n-1}. By (6), $\mathfrak{M}, \langle r_h, x_n \rangle \models \Diamond_h \delta$. So $r_h R_h x_n$ follows, as required. $\Box$

Observe that because of Claim 2.1(iii) and (iv), $\Box_h$ in fact expresses 'horizontal next-time' in our grid. For any formula ψ and any $w \in W_v$,

$$\mathfrak{M}, \langle x_n, w \rangle \models \Box_h \psi \qquad \text{iff} \qquad \mathfrak{M}, \langle x_{n+1}, w \rangle \models \psi, \quad \text{for all } n < \omega. \tag{8}$$

Using this, we will force a pair of infinite lossy and insertion-error $\vec{\tau}$-runs, for the same sequence $\vec{\tau}$ of instructions. Given any counter machine M, for each $i < N$ of its counters, we take two fresh propositional variables C_i° and $\mathsf{C}_i^{\bullet}$. At each moment n of time, the actual content of counter c_i during the lossy run will be represented by the set of points

$$\Sigma_i^{\circ}(n) := \{w \in W_v : x_0 R_v^+ w \text{ and } \mathfrak{M}, \langle x_n, w \rangle \models \mathsf{C}_i^{\circ}\},$$

and during the insertion-error run by the set of points

$$\Sigma_i^{\bullet}(n) := \{w \in W_v : x_0 R_v^+ w \text{ and } \mathfrak{M}, \langle x_n, w \rangle \models \mathsf{C}_i^{\bullet}\}.$$

For each $i < N$, the following formulas force the possible changes in the counters during the lossy and insertion-error runs, respectively:

$$\begin{aligned}
\mathsf{fix}_i^{\circ} &:= \Box_v^+ (\Box_h \mathsf{C}_i^{\circ} \to \mathsf{C}_i^{\circ}), \\
\mathsf{inc}_i^{\circ} &:= \Box_v^+ \big(\Box_h \mathsf{C}_i^{\circ} \to (\mathsf{C}_i^{\circ} \vee \delta)\big), \\
\mathsf{dec}_i^{\circ} &:= \Box_v^+ (\Box_h \mathsf{C}_i^{\circ} \to \mathsf{C}_i^{\circ}) \wedge \Diamond_v^+ (\mathsf{C}_i^{\circ} \wedge \Box_h \neg \mathsf{C}_i^{\circ}),
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{fix}_i^\bullet &:= \Box_v^+(\mathsf{C}_i^\bullet \to \Box_h \mathsf{C}_i^\bullet),\\
\mathsf{inc}_i^\bullet &:= \Box_v^+(\mathsf{C}_i^\bullet \to \Box_h \mathsf{C}_i^\bullet) \wedge \Diamond_v^+(\neg \mathsf{C}_i^\bullet \wedge \Box_h \mathsf{C}_i^\bullet),\\
\mathsf{dec}_i^\bullet &:= \Box_v^+\big(\mathsf{C}_i^\bullet \to (\Box_h \mathsf{C}_i^\bullet \vee \delta)\big).
\end{aligned}$$

Claim 2.2. (lossy and insertion-error counting)
Suppose that $\mathfrak{M}, \langle r_h, r_v\rangle \models \mathsf{grid}$. Then for all $n < \omega$ and $i < N$:

(i) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{fix}_i^\circ$ then $\Sigma_i^\circ(n+1) \subseteq \Sigma_i^\circ(n)$.*

(ii) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{inc}_i^\circ$ then $\Sigma_i^\circ(n+1) \subseteq \Sigma_i^\circ(n) \cup \{x_n\}$.*

(iii) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{dec}_i^\circ$ then $\Sigma_i^\circ(n+1) \subseteq \Sigma_i^\circ(n) - \{z\}$ for some $z \in \Sigma_i^\circ(n)$.*

(iv) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{fix}_i^\bullet$ then $\Sigma_i^\bullet(n+1) \supseteq \Sigma_i^\bullet(n)$.*

(v) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{inc}_i^\bullet$ then there is z such that $x_0 R_v^+ z$, $z \notin \Sigma_i^\bullet(n)$, and $\Sigma_i^\bullet(n+1) \supseteq \Sigma_i^\bullet(n) \cup \{z\}$.*

(vi) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{dec}_i^\bullet$ then $\Sigma_i^\bullet(n+1) \supseteq \Sigma_i^\bullet(n) - \{x_n\}$.*

Proof. We show items (ii) and(v). The proofs of the other items are similar and left to the reader.

(ii): Suppose $w \in \Sigma_i^\circ(n+1)$. Then $x_0 R_v^+ w$ and $\mathfrak{M}, \langle x_{n+1}, w\rangle \models \mathsf{C}_i^\circ$. By (8), we have $\mathfrak{M}, \langle x_n, w\rangle \models \Box_h \mathsf{C}_i^\circ$. Therefore, $\mathfrak{M}, \langle x_n, w\rangle \models \mathsf{C}_i^\circ \vee \delta$ by inc_i°, and so either $w \in \Sigma_i^\circ(n)$ or $w = x_n$.

(v): By $\mathsf{inc}_i^\bullet$, there is z with $x_0 R_v^+ z$ and $\mathfrak{M}, \langle x_n, z\rangle \models \neg \mathsf{C}_i^\bullet \wedge \Box_h \mathsf{C}_i^\bullet$. Thus $z \notin \Sigma_i^\bullet(n)$. Also, we have $\mathfrak{M}, \langle x_{n+1}, z\rangle \models \mathsf{C}_i^\bullet$ by (8), and so $z \in \Sigma_i^\bullet(n+1)$. Now suppose $w \in \Sigma_i^\bullet(n)$. Then $x_0 R_v^+ w$ and $\mathfrak{M}, \langle x_n, w\rangle \models \mathsf{C}_i^\bullet$. By $\mathsf{inc}_i^\bullet$, we have $\mathfrak{M}, \langle x_n, w\rangle \models \Box_h \mathsf{C}_i^\bullet$. Thus $\mathfrak{M}, \langle x_{n+1}, w\rangle \models \mathsf{C}_i^\bullet$ by (8), and so $w \in \Sigma_i^\bullet(n+1)$. ☐

Using the above counting machinery, we can encode lossy and insertion-error steps. For each $\alpha \in Op_C$, we define

$$\mathsf{do}^\circ(\alpha) := \begin{cases} \mathsf{inc}_i^\circ \wedge \bigwedge_{i \neq j < N} \mathsf{fix}_j^\circ, & \text{if } \alpha = c_i^{++},\\ \mathsf{dec}_i^\circ \wedge \bigwedge_{i \neq j < N} \mathsf{fix}_j^\circ, & \text{if } \alpha = c_i^{--},\\ \Box_v^+ \Box_h \neg \mathsf{C}_i^\circ \wedge \bigwedge_{i \neq j < N} \mathsf{fix}_j^\circ, & \text{if } \alpha = c_i^{??}, \end{cases}$$

and

$$\mathsf{do}^\bullet(\alpha) := \begin{cases} \mathsf{inc}_i^\bullet \wedge \bigwedge_{i \neq j < N} \mathsf{fix}_j^\bullet, & \text{if } \alpha = c_i^{++},\\ \mathsf{dec}_i^\bullet \wedge \bigwedge_{i \neq j < N} \mathsf{fix}_j^\bullet, & \text{if } \alpha = c_i^{--},\\ \Box_v^+ \neg \mathsf{C}_i^\bullet \wedge \bigwedge_{i \neq j < N} \mathsf{fix}_j^\bullet, & \text{if } \alpha = c_i^{??}. \end{cases}$$

Now we can force runs of M that start at q_{ini} with all-zero counters. For each state $q \in Q$, we introduce a fresh propositional variable S_q, and define

$$\widehat{\mathsf{S}}_q := \mathsf{S}_q \wedge \bigwedge_{q \neq q' \in Q} \neg \mathsf{S}_{q'}. \tag{9}$$

Let φ_M be the conjunction of

$$\Box_h \Big(\delta \to \big(\widehat{\mathsf{S}}_{q_{\mathsf{ini}}} \wedge \Box_v^+ (\neg \mathsf{C}_i^\circ \wedge \neg \mathsf{C}_i^\bullet) \big) \Big), \tag{10}$$

$$\Box_h \bigwedge_{q \in Q-H} \Big(\widehat{\mathsf{S}}_q \to \bigvee_{\langle \alpha, q' \rangle \in I_q} \big(\Box_h \widehat{\mathsf{S}}_{q'} \wedge \mathsf{do}^\circ(\alpha) \wedge \mathsf{do}^\bullet(\alpha) \big) \Big), \tag{11}$$

$$\Box_h \bigvee_{q \in Q-H} \widehat{\mathsf{S}}_q. \tag{12}$$

Lemma 2.3. (lossy and insertion-error run-emulation)
Suppose that $\mathfrak{M}, \langle r_h, r_v \rangle \models \mathsf{grid} \wedge \varphi_M$. Let $q_0 := q_{\mathsf{ini}}$, and for all $i < N$, $n < \omega$, let $c_i^\circ(n) := |\Sigma_i^\circ(n)|$ and

$$c_i^\bullet(n) := \begin{cases} c_i^\bullet(n-1) + 1, & \textit{if } \Sigma_i^\bullet(n) \textit{ is infinite,} \\ |\Sigma_i^\bullet(n)|, & \textit{otherwise.} \end{cases}$$

Then there exists an infinite sequence $\vec{\tau} = \big\langle \langle \alpha_n, q_n \rangle : 0 < n < \omega \big\rangle$ of instructions such that

- *$\big\langle \langle q_n, \vec{c}^{\,\circ}(n) \rangle : n < \omega \big\rangle$ is a lossy $\vec{\tau}$-run of M, and*
- *$\big\langle \langle q_n, \vec{c}^{\,\bullet}(n) \rangle : n < \omega \big\rangle$ is an insertion-error $\vec{\tau}$-run of M.*

Proof. We define $\big\langle \langle \alpha_n, q_n \rangle : 0 < n < \omega \big\rangle$ by induction on n such that for all $0 < n < \omega$,

- $q_n \in Q - H$ and $\mathfrak{M}, \langle x_n, x_0 \rangle \models \widehat{\mathsf{S}}_{q_n}$,
- $\langle q_{n-1}, \vec{c}^{\,\circ}(n-1) \rangle \to_{lossy}^{\alpha_n} \langle q_n, \vec{c}^{\,\circ}(n) \rangle$ and $\langle q_{n-1}, \vec{c}^{\,\bullet}(n-1) \rangle \to_{i\_err}^{\alpha_n} \langle q_n, \vec{c}^{\,\bullet}(n) \rangle$.

As $\vec{c}^{\,\circ}(0) = \vec{c}^{\,\bullet}(0) = \vec{0}$ by (10), the lemma will follow.

To this end, take some n with $0 < n < \omega$. Then we have $q_{n-1} \in Q - H$ and $\mathfrak{M}, \langle x_{n-1}, x_0 \rangle \models \widehat{\mathsf{S}}_{q_{n-1}}$, by (10) and (12) if $n = 1$, and by the IH if $n > 1$. Therefore, by Claim 2.1(i) and (11), there is $\langle \alpha_n, q_n \rangle \in I_{q_{n-1}}$ such that $\mathfrak{M}, \langle x_{n-1}, x_0 \rangle \models \Box_h \widehat{\mathsf{S}}_{q_n} \wedge \mathsf{do}^\circ(\alpha_n) \wedge \mathsf{do}^\bullet(\alpha_n)$. So $\mathfrak{M}, \langle x_n, x_0 \rangle \models \widehat{\mathsf{S}}_{q_n}$ by Claim 2.1(iii), and so $q_n \in Q - H$ by Claim 2.1(i) and (12). Using Claim 2.2(i)–(iii), it is easy to check that $\langle q_{n-1}, \vec{c}^{\,\circ}(n-1) \rangle \to_{lossy}^{\alpha_n} \langle q_n, \vec{c}^{\,\circ}(n) \rangle$. Finally, in order to show that $\langle q_{n-1}, \vec{c}^{\,\bullet}(n-1) \rangle \to_{i\_err}^{\alpha_n} \langle q_n, \vec{c}^{\,\bullet}(n) \rangle$, we need to use Claim 2.2(iv)–(vi) and the following observation. As for each $i < N$ either $\Sigma_i^\bullet(n-1)$ is infinite or $c_i^\bullet(n-1) = |\Sigma_i^\bullet(n-1)|$, if $c_i^\bullet(n-1) \neq 0$ then $\Sigma_i^\bullet(n-1) \neq \emptyset$, and so $\alpha_n \neq c_i^{??}$ follows by $\mathfrak{M}, \langle x_{n-1}, x_0 \rangle \models \mathsf{do}^\bullet(\alpha_n)$. ☐

For each $k \leq \omega$, let $\mathfrak{H}_k$ be the frame obtained from $\langle k, +1 \rangle$ by adding a 'spy-point', that is, let $\mathfrak{H}_k := \langle k+1, S_k \rangle$, where

$$S_k = \{ \langle k, n \rangle : n < k \} \cup \{ \langle n-1, n \rangle : 0 < n < k \}. \tag{13}$$

Lemma 2.4. (soundness)
If M has an infinite reliable run, then $\mathsf{grid} \wedge \varphi_M$ is satisfiable in a model over $\mathfrak{H}_\omega \times^\delta \mathfrak{F}$ for some ω-fan $\mathfrak{F}$.

Proof. Suppose that $\big\langle \langle q_n, \vec{c}(n)\rangle : n < \omega \big\rangle$ is a reliable $\vec{\tau}$-run of M, for some sequence $\vec{\tau} = \big\langle \langle \alpha_n, q_n\rangle : 0 < n < \omega \big\rangle$ of instructions. We define a model $\mathfrak{M}_\infty = \langle \mathfrak{H}_\omega \times^\delta \mathfrak{F}, \mu\rangle$ as follows. For each $q \in Q$, we let

$$\mu(\mathsf{S}_q) := \{\langle n, 0\rangle : n < \omega,\ q_n = q\}.$$

Further, for all $i < N$, $n < \omega$, we will define inductively the sets $\mu_n(\mathsf{C}_i^\circ)$ and $\mu_n(\mathsf{C}_i^\bullet)$, and then put

$$\mu(\mathsf{C}_i^\circ) := \{\langle n, m\rangle : m \in \mu_n(\mathsf{C}_i^\circ)\} \ \text{ and } \ \mu(\mathsf{C}_i^\bullet) := \{\langle n, m\rangle : m \in \mu_n(\mathsf{C}_i^\bullet)\}.$$

To begin with, we let $\mu_0(\mathsf{C}_i^\circ) = \mu_0(\mathsf{C}_i^\bullet) := \emptyset$, and

$$\mu_{n+1}(\mathsf{C}_i^\circ) := \begin{cases} \mu_n(\mathsf{C}_i^\circ) \cup \{n\}, & \text{if } \alpha_{n+1} = c_i^{++}, \\ \mu_n(\mathsf{C}_i^\circ) - \{\min \mu_n(\mathsf{C}_i^\circ)\}, & \text{if } \alpha_{n+1} = c_i^{--}, \\ \mu_n(\mathsf{C}_i^\circ), & \text{otherwise.} \end{cases}$$

It is straightforward to check that

$$|\mu_n(\mathsf{C}_i^\circ)| = c_i(n) \ \text{ and } \ \mathfrak{M}_\infty, \langle n, 0\rangle \models \mathsf{do}^\circ(\alpha_{n+1}), \quad \text{for all } i < N,\ n < \omega. \tag{14}$$

We need to be a bit more careful when defining $\mu_n(\mathsf{C}_i^\bullet)$. As the formulas $\mathsf{do}^\bullet(\alpha_n)$ permit decrementing the insertion-error counters only at diagonal points, we must be sure that only previously incremented points get decremented. To this end, for every $i < N$, we let

$$\Lambda_i := \{k < \omega : \alpha_{k+1} = c_i^{--}\}, \qquad \Xi_i := \{k < \omega : \alpha_{k+1} = c_i^{++}\}, \tag{15}$$

and let

$$\langle \lambda_m^i : m < L_i\rangle \text{ be the enumeration of } \Lambda_i \text{ in ascending order, and} \tag{16}$$

$$\langle \xi_m^i : m < K_i\rangle \text{ be the enumeration of } \Xi_i \text{ in ascending order,} \tag{17}$$

for some $L_i, K_i \le \omega$. As in a run only non-zero counters can be decremented and our run is reliable, we always have $L_i \le K_i$, and $\lambda_m^i > \xi_m^i$ for all $m < L_i$. Then we let

$$\mu_{n+1}(\mathsf{C}_i^\bullet) := \begin{cases} \mu_n(\mathsf{C}_i^\bullet) \cup \{\lambda_m^i\}, & \text{if } \alpha_{n+1} = c_i^{++},\ n = \xi_m^i, \\ & \qquad m < L_i, \\ \mu_n(\mathsf{C}_i^\bullet) \cup \big\{\min\big(\omega - \mu_n(\mathsf{C}_i^\bullet)\big)\big\}, & \text{if } \alpha_{n+1} = c_i^{++},\ n = \xi_m^i, \\ & \qquad L_i \le m < K_i, \\ \mu_n(\mathsf{C}_i^\bullet) - \{n\}, & \text{if } \alpha_{n+1} = c_i^{--}, \\ \mu_n(\mathsf{C}_i^\bullet), & \text{otherwise.} \end{cases}$$

We claim that if $\alpha_{n+1} = c_i^{--}$ then $n \in \mu_n(\mathsf{C}_i^\bullet)$, and so $|\mu_{n+1}(\mathsf{C}_i^\bullet)| = |\mu_n(\mathsf{C}_i^\bullet)| - 1$. Indeed, if $\alpha_{n+1} = c_i^{--}$ then $n = \lambda_m^i$ for some $m < L_i$. So $\mu_{\xi_m^i+1}(\mathsf{C}_i^\bullet) = \mu_{\xi_m^i}(\mathsf{C}_i^\bullet) \cup \{\lambda_m^i\}$, and so $n \in \mu_{\xi_m^i+1}(\mathsf{C}_i^\bullet)$. It follows that $n \in \mu_k(\mathsf{C}_i^\bullet)$ for every k with $\xi_m^i + 1 \le k < n+1$, as required.

Now it is not hard to see that $|\mu_n(\mathsf{C}_i^\bullet)| = c_i(n)$ and $\mathfrak{M}_\infty, \langle n, 0\rangle \models \mathsf{do}^\bullet(\alpha_{n+1})$, for all $i < N$ and $n < \omega$. Using this and (14), it is easy to check that $\mathfrak{M}_\infty, \langle \omega, 0\rangle \models \mathsf{grid} \wedge \varphi_M$. $\square$

Now Theorem 2 follows from Prop. 3, Lemmas 2.3 and 2.4.

Note that it is easy to generalise the proof to obtain undecidability of $\mathbf{T} \times^\delta L$ (where $\mathbf{T}$ is the unimodal logic of all reflexive frames), by using a version of the 'tick-' or 'chessboard'-trick (see e.g. [39, 32, 10] for more details): Take a fresh propositional variable tick, and define a new 'horizontal' modal operator by setting, for all formulas ϕ,

$$\blacksquare_h \phi := \big(\mathsf{tick} \to \Box_h(\neg\mathsf{tick} \to \phi)\big) \wedge \big(\neg\mathsf{tick} \to \Box_h(\mathsf{tick} \to \phi)\big). \tag{18}$$

Then replace each occurrence of $\Box_h$ in the formula $\mathsf{grid} \wedge \varphi_M$ with $\blacksquare_h$, and add the conjunct

$$\Box_h\big((\mathsf{tick} \leftrightarrow \Box_v\mathsf{tick}) \wedge (\neg\mathsf{tick} \leftrightarrow \Box_v\neg\mathsf{tick})\big). \tag{19}$$

It is not hard to check that the resulting formula is $\mathbf{T} \times^\delta L$-satisfiable iff M has an infinite reliable run.

Next, recall k-fans from (5), and the frames $\mathfrak{H}_k$ from (13).

Theorem 3. *Let $\mathcal{C}_h$ and $\mathcal{C}_v$ be any classes of frames such that*

- *either $\mathcal{C}_h$ or $\mathcal{C}_v$ contains only finite frames,*
- *either $\mathfrak{H}_\omega \in \mathcal{C}_h$, or $\mathfrak{H}_k \in \mathcal{C}_h$ for every $k < \omega$,*
- *either $\mathcal{C}_v$ contains an ω-fan, or $\mathcal{C}_v$ contains a k-fan for every $k < \omega$.*

Then $\mathsf{Logic\_of}(\mathcal{C}_h \times^\delta \mathcal{C}_v)$ *is not recursively enumerable.*

Proof. We sketch how to modify the proof of Theorem 2 to obtain a reduction of the 'CM reachability' problem to $\mathcal{C}_h \times^\delta \mathcal{C}_v$-satisfiability. To begin with, observe that if we add the conjunct

$$\Box_h\Box_v^+\big(p \vee \delta \to \Box_h(p \wedge \neg\delta)\big) \tag{20}$$

to the formula grid defined in (6)–(7), then the grid-points x_n generated in Claim 2.1 are all different. Now we introduce a fresh propositional variable end, and let $\mathsf{grid}^{\mathit{fin}}$ be the conjunction of (6), (20) and the following 'finitary' version of (7):

$$\Box_h\Diamond_v\big(\mathsf{end} \vee (\Diamond_h\delta \wedge \Box_h\delta)\big). \tag{21}$$

Given any counter machine M and a state q_{fin}, let φ_M^{fin} be obtained from φ_M by replacing (12) with

$$\Box_h \bigvee_{q \in (Q-H)\cup\{q_{\mathsf{fin}}\}} \widehat{\mathsf{S}}_q.$$

It is not hard to see that $\mathsf{grid}^{\mathit{fin}} \wedge \varphi_M^{\mathit{fin}} \wedge \Box_h(\Diamond_v\mathsf{end} \to \widehat{\mathsf{S}}_{q_{\mathsf{fin}}})$ is $\mathcal{C}_h \times^\delta \mathcal{C}_v$-satisfiable iff there is a reliable run of M reaching q_{fin}. $\Box$

Note that it is also possible to give another proof of Theorem 2 by doing everything '*backwards*'. The conjunction of the following formulas generates a grid backwards in $\mathbf{K} \times^\delta L$-frames, and is used in [22] to show that these logics lack the finite model property w.r.t. *any* (not necessarily product) frames:

$$\Diamond_v\Diamond_h(\delta \wedge \Box_h\bot),$$
$$\Box_v\big(\Diamond_h\delta \to \Diamond_h(\neg\delta \wedge \Diamond_h\delta \wedge \Box_h\delta)\big),$$
$$\Box_h\Diamond_v\delta.$$

Then the conjunction of the following formulas emulates counter machine runs, again by going backwards along the generated grid:

$$\Box_h\Big(\Box_h\bot \to \big(\widehat{\mathsf{S}}_{q_{\mathsf{ini}}} \wedge \Box_v^+(\neg\mathsf{C}_i^\circ \wedge \neg\mathsf{C}_i^\bullet)\big)\Big),$$

$$\Box_h \bigwedge_{q\in Q-H}\Big(\Diamond_h\widehat{\mathsf{S}}_q \to \bigvee_{\langle\alpha,q'\rangle\in I_q}\big(\widehat{\mathsf{S}}_{q'} \wedge \mathsf{bw\_do}^\circ(\alpha)\wedge\mathsf{bw\_do}^\bullet(\alpha)\big)\Big),$$

$$\Box_h \bigvee_{q\in Q-H} \widehat{\mathsf{S}}_q,$$

where

$$\mathsf{bw\_do}^\circ(\alpha) := \begin{cases} \mathsf{bw\_inc}_i^\circ \wedge \bigwedge_{i\neq j<N}\mathsf{bw\_fix}_j^\circ, & \text{if } \alpha = c_i^{++},\\ \mathsf{bw\_dec}_i^\circ \wedge \bigwedge_{i\neq j<N}\mathsf{bw\_fix}_j^\circ, & \text{if } \alpha = c_i^{--},\\ \Box_v^+\neg\mathsf{C}_i^\circ \wedge \bigwedge_{i\neq j<N}\mathsf{bw\_fix}_j^\circ, & \text{if } \alpha = c_i^{??},\end{cases}$$

$$\mathsf{bw\_do}^\bullet(\alpha) := \begin{cases} \mathsf{bw\_inc}_i^\bullet \wedge \bigwedge_{i\neq j<N}\mathsf{bw\_fix}_j^\bullet, & \text{if } \alpha = c_i^{++},\\ \mathsf{bw\_dec}_i^\bullet \wedge \bigwedge_{i\neq j<N}\mathsf{bw\_fix}_j^\bullet, & \text{if } \alpha = c_i^{--},\\ \Box_v^+\Box_h\neg\mathsf{C}_i^\bullet \wedge \bigwedge_{i\neq j<N}\mathsf{bw\_fix}_j^\bullet, & \text{if } \alpha = c_i^{??},\end{cases}$$

$$\begin{aligned}
\mathsf{bw_fix}_i^\circ &:= \Box_v^+(\mathsf{C}_i^\circ \to \Box_h\mathsf{C}_i^\circ),\\
\mathsf{bw_inc}_i^\circ &:= \Box_v^+\big(\mathsf{C}_i^\circ \to (\Box_h\mathsf{C}_i^\circ \vee \delta)\big),\\
\mathsf{bw_dec}_i^\circ &:= \Box_v^+(\mathsf{C}_i^\circ \to \Box_h\mathsf{C}_i^\circ) \wedge \Diamond_v^+(\neg\mathsf{C}_i^\circ \wedge \Box_h\mathsf{C}_i^\circ),\\
\mathsf{bw_fix}_i^\bullet &:= \Box_v^+(\Box_h\mathsf{C}_i^\bullet \to \mathsf{C}_i^\bullet),\\
\mathsf{bw_inc}_i^\bullet &:= \Box_v^+(\Box_h\mathsf{C}_i^\bullet \to \mathsf{C}_i^\bullet) \wedge \Diamond_v^+(\mathsf{C}_i^\bullet \wedge \Box_h\neg\mathsf{C}_i^\bullet),\\
\mathsf{bw_dec}_i^\bullet &:= \Box_v^+\big(\Box_h\mathsf{C}_i^\bullet \to (\mathsf{C}_i^\bullet \vee \delta)\big),
\end{aligned}$$

for $i < N$.

6 Undecidable δ-products with a 'linear' component

Theorem 4. *Let L_h be any Kripke complete logic such that L_h contains* **K4.3** *and $\langle\omega,<\rangle$ is a frame for L_h. Let L_v be any Kripke complete logic having an ω-fan among its frames. Then $L_h \times^\delta L_v$ is undecidable.*

Corollary 3. **K4.3** $\times^\delta$ **S5** *and* **K4.3** $\times^\delta$ **K** *are both undecidable.*

We prove Theorem 4 by reducing the 'CM non-termination' problem to $L_h \times^\delta L_v$-satisfiability. Let $\mathfrak{M}$ be a model based on the δ-product of a frame $\mathfrak{F}_h = \langle W_h, R_h\rangle$ for L_h (so R_h is transitive and weakly connected[2]), and some frame $\mathfrak{F}_v = \langle W_v, R_v\rangle$ for L_v. First, we again generate an $\omega\times\omega$-grid in $\mathfrak{M}$. Let

$$\mathsf{lingrid} := \ \delta \wedge \Box_h^+\Diamond_v(\Diamond_h\delta \wedge \Box_h\Box_h\neg\delta).$$

[2] A relation R is called *weakly connected* if $\forall x,y,z\,\big(xRy \wedge xRz \to (y = z \vee yRz \vee zRy)\big)$.

Claim 4.1. (grid generation)
If $\mathfrak{M}, \langle r_h, r_v\rangle \models \mathsf{lingrid}$ *then there exist points* $\langle x_n \in W_h \cap W_v : n < \omega\rangle$ *such that, for all* $n < \omega$,

(i) $x_0 = r_v$, *and if* $n > 0$ *then* $x_0 R_v x_n$,

(ii) *if* $n > 0$ *then* $\mathfrak{M}, \langle x_{n-1}, x_n\rangle \models \Diamond_h \delta \land \Box_h \Box_h \neg\delta$,

(iii) *if* $n > 0$ *then, for every* z, $x_{n-1} R_h z$ *implies that* $z = x_n$ *or* $x_n R_h z$,

(iv) $x_0 = r_h$ *and* $x_m R_h x_n$ *for all* $m < n$.

Proof. By induction on n. Let $x_0 := r_h$. As $\mathfrak{M}, \langle r_h, r_v\rangle \models \delta$, we have $r_h = r_v$. Now suppose inductively that we have $\langle x_k : k < n\rangle$ satisfying (i)–(iv) for some $0 < n < \omega$. Then there is $x_n \in W_v$ such that $x_0 R_v x_n$ and $\mathfrak{M}, \langle x_{n-1}, x_n\rangle \models \Diamond_h \delta \land \Box_h \Box_h \neg\delta$. Therefore, $x_n \in W_h$, $x_{n-1} R_h x_n$, and for every z, $x_{n-1} R_h z$ implies that $z = x_n$ or $x_n R_h z$, by the weak connectedness of R_h. So by the IH and the transitivity of R_h, we have $x_m R_h x_n$ for all $m < n$. $\Box$

Next, given any counter machine M, we will again force both an infinite lossy and an infinite insertion-error $\vec{\tau}$-run, for the same sequence $\vec{\tau}$ of instructions. As R_h is transitive, we do not have a general 'horizontal next-time' operator in our grid, like we had in (8). However, because of Claim 4.1(iii) and (iv), we still can have the following: For any formula ψ and any $w \in W_v$,

$$\text{if } \psi \text{ is such that } \mathfrak{M}, \langle x_{n+1}, w\rangle \models \psi \to \Box_h \psi, \text{ then}$$
$$\mathfrak{M}, \langle x_n, w\rangle \models \Box_h \psi \quad \text{iff} \quad \mathfrak{M}, \langle x_{n+1}, w\rangle \models \psi, \quad \text{for all } n < \omega. \tag{22}$$

In order to utilise this, for each counter $i < N$ of M, we introduce two pairs of propositional variables: In_i°, Out_i° for emulating lossy behaviour, and $\mathsf{In}_i^\bullet$, $\mathsf{Out}_i^\bullet$ for emulating insertion-error behaviour. The following formula ensures that the condition in (22) hold for each of these variables, at all the relevant points in $\mathfrak{M}$:

$$\xi_M := \bigwedge_{i<N} \Box_h^+ \Box_v^+ \big((\mathsf{In}_i^\circ \to \Box_h \mathsf{In}_i^\circ) \land (\mathsf{Out}_i^\circ \to \Box_h \mathsf{Out}_i^\circ) \land (\mathsf{In}_i^\bullet \to \Box_h \mathsf{In}_i^\bullet) \land (\mathsf{Out}_i^\bullet \to \Box_h \mathsf{Out}_i^\bullet) \big).$$

At each moment n of time, the actual content of counter c_i during the lossy run will be represented by the set of points

$$\Delta_i^\circ(n) := \{w \in W_v : x_0 R_v^+ w \text{ and } \mathfrak{M}, \langle x_n, w\rangle \models \mathsf{In}_i^\circ \land \neg\mathsf{Out}_i^\circ\},$$

and during the insertion-error run by the set of points

$$\Delta_i^\bullet(n) := \{w \in W_v : x_0 R_v^+ w \text{ and } \mathfrak{M}, \langle x_n, w\rangle \models \mathsf{In}_i^\bullet \land \neg\mathsf{Out}_i^\bullet\}.$$

For each $i < N$, the following formulas force the possible changes in the counters during the lossy and insertion-error runs, respectively:

$$\begin{aligned}
\mathsf{lin_fix}_i^\circ &:= \Box_v^+ (\Box_h \mathsf{In}_i^\circ \to \mathsf{In}_i^\circ), \\
\mathsf{lin_inc}_i^\circ &:= \Box_v^+ \big(\Box_h \mathsf{In}_i^\circ \to (\mathsf{In}_i^\circ \lor \delta)\big), \\
\mathsf{lin_dec}_i^\circ &:= \Box_v^+ (\Box_h \mathsf{In}_i^\circ \to \mathsf{In}_i^\circ) \land \Diamond_v^+ (\mathsf{In}_i^\circ \land \neg\mathsf{Out}_i^\circ \land \Box_h \mathsf{Out}_i^\circ),
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{lin_fix}_i^\bullet &:= \Box_v^+(\Box_h\mathsf{Out}_i^\bullet \to \mathsf{Out}_i^\bullet),\\
\mathsf{lin_inc}_i^\bullet &:= \Box_v^+(\Box_h\mathsf{Out}_i^\bullet \to \mathsf{Out}_i^\bullet) \wedge \Diamond_v^+(\neg\mathsf{In}_i^\bullet \wedge \neg\mathsf{Out}_i^\bullet \wedge \Box_h\mathsf{In}_i^\bullet),\\
\mathsf{lin_dec}_i^\bullet &:= \Box_v^+\big(\Box_h\mathsf{Out}_i^\bullet \to (\mathsf{Out}_i^\bullet \vee \delta)\big).
\end{aligned}$$

Claim 4.2. (lossy and insertion-error counting)
Suppose that $\mathfrak{M}, \langle r_h, r_v\rangle \models \mathsf{lingrid} \wedge \xi_M$. Then for all $n < \omega$, $i < N$:

(i) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{lin\_fix}_i^\circ$ then $\Delta_i^\circ(n+1) \subseteq \Delta_i^\circ(n)$.*

(ii) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{lin\_inc}_i^\circ$ then $\Delta_i^\circ(n+1) \subseteq \Delta_i^\circ(n) \cup \{x_n\}$.*

(iii) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{lin\_dec}_i^\circ$ then $\Delta_i^\circ(n+1) \subseteq \Delta_i^\circ(n) - \{z\}$ for some $z \in \Delta_i^\circ(n)$.*

(iv) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{lin\_fix}_i^\bullet$ then $\Delta_i^\bullet(n+1) \supseteq \Delta_i^\bullet(n)$.*

(v) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{lin\_inc}_i^\bullet$ then there is z such that $x_0 R_v^+ z$, $z \notin \Delta_i^\bullet(n)$, and $\Delta_i^\bullet(n+1) \supseteq \Delta_i^\bullet(n) \cup \{z\}$.*

(vi) *If $\mathfrak{M}, \langle x_n, x_0\rangle \models \mathsf{lin\_dec}_i^\bullet$ then $\Delta_i^\bullet(n+1) \supseteq \Delta_i^\bullet(n) - \{x_n\}$.*

Proof. We show items (iii) and (vi). The proofs of the other items are similar and left to the reader.

(iii): By $\mathsf{lin\_dec}_i^\circ$, there is z such that $x_0 R_v^+ z$ and

$$\mathfrak{M}, \langle x_n, z\rangle \models \mathsf{In}_i^\circ \wedge \neg\mathsf{Out}_i^\circ \wedge \Box_h\mathsf{Out}_i^\circ.$$

So $z \in \Delta_i^\circ(n)$. Also, by Claim 4.1(iv),

$$\mathfrak{M}, \langle x_{n+1}, z\rangle \models \mathsf{Out}_i^\circ. \tag{23}$$

Now suppose $w \in \Delta_i^\circ(n+1)$. Then $x_0 R_v^+ w$ and $\mathfrak{M}, \langle x_{n+1}, w\rangle \models \mathsf{In}_i^\circ \wedge \neg\mathsf{Out}_i^\circ$. Then $\mathfrak{M}, \langle x_n, w\rangle \models \neg\mathsf{Out}_i^\circ$ by ξ_M and Claim 4.1(iv), and $\mathfrak{M}, \langle x_n, w\rangle \models \Box_h\mathsf{In}_i^\circ$ by ξ_M and (22). So we have $\mathfrak{M}, \langle x_n, w\rangle \models \mathsf{In}_i^\circ$ by $\mathsf{lin\_dec}_i^\circ$, and so $w \in \Delta_i^\circ(n)$. Finally, $w \neq z$ by (23).

(vi): Suppose that $w \in \Delta_i^\bullet(n) - \{x_n\}$. Then $x_0 R_v^+ w$ and $\mathfrak{M}, \langle x_n, w\rangle \models \mathsf{In}_i^\bullet \wedge \neg\mathsf{Out}_i^\bullet \wedge \neg\delta$. Then $\mathfrak{M}, \langle x_{n+1}, w\rangle \models \mathsf{In}_i^\bullet$ by ξ_M and Claim 4.1(iv), and $\mathfrak{M}, \langle x_n, w\rangle \models \neg\Box_h\mathsf{Out}_i^\bullet$ by $\mathsf{lin\_dec}_i^\bullet$. Therefore, $\mathfrak{M}, \langle x_{n+1}, w\rangle \models \neg\mathsf{Out}_i^\bullet$ by ξ_M and (22), and so we have $w \in \Delta_i^\bullet(n+1)$. ◻

For each $\alpha \in Op_C$, we define

$$\mathsf{lin_do}^\circ(\alpha) := \begin{cases}
\mathsf{lin_inc}_i^\circ \wedge \displaystyle\bigwedge_{i\neq j<N} \mathsf{lin_fix}_j^\circ, & \text{if } \alpha = c_i^{++},\\
\mathsf{lin_dec}_i^\circ \wedge \displaystyle\bigwedge_{i\neq j<N} \mathsf{lin_fix}_j^\circ, & \text{if } \alpha = c_i^{--},\\
\Box_v^+(\Box_h\mathsf{In}_i^\circ \to \Box_h\mathsf{Out}_i^\circ) \wedge \displaystyle\bigwedge_{i\neq j<N} \mathsf{lin_fix}_j^\circ, & \text{if } \alpha = c_i^{??},
\end{cases}$$

and

$$\mathsf{lin_do}^\bullet(\alpha) := \begin{cases}
\mathsf{lin_inc}_i^\bullet \wedge \displaystyle\bigwedge_{i\neq j<N} \mathsf{lin_fix}_j^\bullet, & \text{if } \alpha = c_i^{++},\\
\mathsf{lin_dec}_i^\bullet \wedge \displaystyle\bigwedge_{i\neq j<N} \mathsf{lin_fix}_j^\bullet, & \text{if } \alpha = c_i^{--},\\
\Box_v^+(\mathsf{In}_i^\bullet \to \mathsf{Out}_i^\bullet) \wedge \displaystyle\bigwedge_{i\neq j<N} \mathsf{lin_fix}_j^\bullet, & \text{if } \alpha = c_i^{??}.
\end{cases}$$

For each state $q \in Q$, we introduce a fresh propositional variable S_q, and define the formula $\widehat{\mathsf{S}}_q$ as in (9). Let ψ_M be the conjunction of ξ_M and the following formulas:

$$\widehat{\mathsf{S}}_{q_{\mathsf{ini}}} \wedge \Box_v^+(\neg\mathsf{In}_i^\circ \wedge \neg\mathsf{Out}_i^\circ \wedge \neg\mathsf{In}_i^\bullet \wedge \neg\mathsf{Out}_i^\bullet), \tag{24}$$

$$\Box_h^+ \bigwedge_{q\in Q-H} \Big[\Diamond_v^+\widehat{\mathsf{S}}_q \to \bigvee_{\langle\alpha,q'\rangle\in I_q} \Big(\mathsf{lin\_do}^\circ(\alpha) \wedge \mathsf{lin\_do}^\bullet(\alpha) \wedge \Box_v^+\big(\Diamond_h\delta \wedge \Box_h\Box_h\neg\delta \to \Box_h(\delta \to \widehat{\mathsf{S}}_{q'})\big)\Big)\Big], \tag{25}$$

$$\Box_h^+\Box_v^+\big(\delta \to \bigvee_{q\in Q-H} \widehat{\mathsf{S}}_q\big). \tag{26}$$

Lemma 4.3. (lossy and insertion-error run-emulation)
Suppose that $\mathfrak{M}, \langle r_h, r_v\rangle \models \mathsf{lingrid} \wedge \psi_M$. Let $q_0 := q_{\mathsf{ini}}$, and for all $i < N$, $n < \omega$, let $c_i^\circ(n) := |\Delta_i^\circ(n)|$ and

$$c_i^\bullet(n) := \begin{cases} c_i^\bullet(n-1)+1, & \textit{if } \Delta_i^\bullet(n) \textit{ is infinite,} \\ |\Delta_i^\bullet(n)|, & \textit{otherwise.} \end{cases}$$

Then there exists an infinite sequence $\vec{\tau} = \big\langle\langle\alpha_n, q_n\rangle : 0 < n < \omega\big\rangle$ of instructions such that

- *$\big\langle\langle q_n, \vec{c}^{\,\circ}(n)\rangle : n < \omega\big\rangle$ is a lossy $\vec{\tau}$-run of M, and*
- *$\big\langle\langle q_n, \vec{c}^{\,\bullet}(n)\rangle : n < \omega\big\rangle$ is an insertion-error $\vec{\tau}$-run of M.*

Proof. We define $\big\langle\langle\alpha_n, q_n\rangle : 0 < n < \omega\big\rangle$ by induction on n such that for all $0 < n < \omega$

- $q_n \in Q - H$ and $\mathfrak{M}, \langle x_n, x_n\rangle \models \widehat{\mathsf{S}}_{q_n}$,
- $\langle q_{n-1}, \vec{c}^{\,\circ}(n-1)\rangle \to^{\alpha_n}_{lossy} \langle q_n, \vec{c}^{\,\circ}(n)\rangle$ and $\langle q_{n-1}, \vec{c}^{\,\bullet}(n-1)\rangle \to^{\alpha_n}_{i\_err} \langle q_n, \vec{c}^{\,\bullet}(n)\rangle$.

As $\vec{c}^{\,\circ}(0) = \vec{c}^{\,\bullet}(0) = \vec{0}$ by (24), the lemma will follow.

To this end, take some n with $0 < n < \omega$. Then we have $q_{n-1} \in Q-H$ and $\mathfrak{M}, \langle x_{n-1}, x_{n-1}\rangle \models \widehat{\mathsf{S}}_{q_{n-1}}$, by (24) and (26) if $n = 1$, and by the IH if $n > 1$. So by Claim 4.1(i), we have $\mathfrak{M}, \langle x_{n-1}, x_0\rangle \models \Diamond_v^+\widehat{\mathsf{S}}_{q_{n-1}}$. Thus by Claim 4.1(iv) and (25), there is $\langle\alpha_n, q_n\rangle \in I_{q_{n-1}}$ such that $\mathfrak{M}, \langle x_{n-1}, x_0\rangle \models \mathsf{lin\_do}^\circ(\alpha_n) \wedge \mathsf{lin\_do}^\bullet(\alpha_n)$ and

$$\mathfrak{M}, \langle x_{n-1}, x_0\rangle \models \Box_v^+\big(\Diamond_h\delta \wedge \Box_h\Box_h\neg\delta \to \Box_h(\delta \to \widehat{\mathsf{S}}_{q'})\big). \tag{27}$$

Now it is easy to check that $\langle q_{n-1}, \vec{c}^{\,\circ}(n-1)\rangle \to^{\alpha_n}_{lossy} \langle q_n, \vec{c}^{\,\circ}(n)\rangle$ holds, using Claim 4.2(i)–(iii). In order to show that $\langle q_{n-1}, \vec{c}^{\,\bullet}(n-1)\rangle \to^{\alpha_n}_{i\_err} \langle q_n, \vec{c}^{\,\bullet}(n)\rangle$, we need to use Claim 4.2(iv)–(vi) and the following observation. As for each $i < N$ either $\Delta_i^\bullet(n-1)$ is infinite or $c_i^\bullet(n-1) = |\Delta_i^\bullet(n-1)|$, if $c_i^\bullet(n-1) \neq 0$ then $\Delta_i^\bullet(n-1) \neq \emptyset$, and so $\alpha_n \neq c_i^{??}$ follows by $\mathfrak{M}, \langle x_{n-1}, x_0\rangle \models \mathsf{lin\_do}^\bullet(\alpha_n)$. Finally, we have $\mathfrak{M}, \langle x_n, x_n\rangle \models \widehat{\mathsf{S}}_{q_n}$ by (27) and Claim 4.1(ii),(iv), and so $q_n \in Q - H$ by Claim 4.1(i),(iv) and (26). ☐

Lemma 4.4. (soundness)
If M has an infinite reliable run, then $\mathsf{lingrid} \wedge \psi_M$ is satisfiable in a model over $\langle\omega, <\rangle \times^\delta \mathfrak{F}$ for some countably infinite one-step rooted frame $\mathfrak{F}$.

Proof. We may assume that $\mathfrak{F} = \langle \omega, S\rangle$ and $\{\langle 0,n\rangle : 0 < n < \omega\} \subseteq S$. Suppose that $\big\langle \langle q_n, \vec{c}(n)\rangle : n < \omega \big\rangle$ is a reliable run of M, for some sequence $\vec{\tau} = \big\langle \langle \alpha_n, q_n\rangle : 0 < n < \omega\big\rangle$ of instructions. We define a model

$$\mathfrak{N}_\infty = \big\langle \langle \omega, <\rangle \times^\delta \mathfrak{F}, \nu \big\rangle$$

as follows. For each $q \in Q$, we let

$$\nu(\mathsf{S}_q) := \{\langle n,n\rangle : n < \omega,\ q_n = q\}.$$

Further, for all $i < N$, $n < \omega$, we will define inductively the sets $\nu_n(\mathsf{In}_i^\circ)$, $\nu_n(\mathsf{Out}_i^\circ)$, $\nu_n(\mathsf{In}_i^\bullet)$, and $\nu_n(\mathsf{Out}_i^\bullet)$, and then put

$$\nu(\mathsf{P}) := \{\langle n,m\rangle : m \in \nu_n(\mathsf{P})\},$$

for $\mathsf{P} \in \{\mathsf{In}_i^\circ, \mathsf{Out}_i^\circ, \mathsf{In}_i^\bullet, \mathsf{Out}_i^\bullet\}$. To begin with, we let $\nu_0(\mathsf{In}_i^\circ) = \nu_0(\mathsf{Out}_i^\circ) = \nu_0(\mathsf{In}_i^\bullet) = \nu_0(\mathsf{Out}_i^\bullet) := \emptyset$, and

$$\nu_{n+1}(\mathsf{In}_i^\circ) := \begin{cases} \nu_n(\mathsf{In}_i^\circ) \cup \{n\}, & \text{if } \alpha_{n+1} = c_i^{++},\\ \nu_n(\mathsf{In}_i^\circ), & \text{otherwise,}\end{cases}$$

$$\nu_{n+1}(\mathsf{Out}_i^\circ) := \begin{cases} \nu_n(\mathsf{Out}_i^\circ) \cup \{\min\big(\nu_n(\mathsf{In}_i^\circ) - \nu_n(\mathsf{Out}_i^\circ)\big)\}, & \text{if } \alpha_{n+1} = c_i^{--},\\ \nu_n(\mathsf{Out}_i^\circ), & \text{otherwise,}\end{cases}$$

$$\nu_{n+1}(\mathsf{Out}_i^\bullet) := \begin{cases} \nu_n(\mathsf{Out}_i^\bullet) \cup \{n\}, & \text{if } \alpha_{n+1} = c_i^{--},\\ \nu_n(\mathsf{Out}_i^\bullet), & \text{otherwise.}\end{cases}$$

Next, recall the notation introduced in (15)–(17). We let

$$\nu_{n+1}(\mathsf{In}_i^\bullet) := \begin{cases} \nu_n(\mathsf{In}_i^\bullet) \cup \{\lambda_m^i\}, & \text{if } \alpha_{n+1} = c_i^{++},\ n = \xi_m^i,\\ & \qquad m < L_i,\\ \nu_n(\mathsf{In}_i^\bullet) \cup \big\{\min\big(\omega - \nu_n(\mathsf{In}_i^\bullet)\big)\big\}, & \text{if } \alpha_{n+1} = c_i^{++},\ n = \xi_m^i,\\ & \qquad L_i \le m < K_i,\\ \nu_n(\mathsf{In}_i^\bullet), & \text{otherwise.}\end{cases}$$

We claim that if $\alpha_{n+1} = c_i^{--}$ then $n \in \nu_n(\mathsf{C}_i^\bullet) = \nu_{n+1}(\mathsf{C}_i^\bullet)$, and so

$$|\nu_{n+1}(\mathsf{In}_i^\bullet) - \nu_{n+1}(\mathsf{Out}_i^\bullet)| = |\nu_n(\mathsf{In}_i^\bullet) - \nu_n(\mathsf{Out}_i^\bullet)| - 1.$$

Indeed, if $\alpha_{n+1} = c_i^{--}$ then $n = \lambda_m^i$ for some $m < L_i$. So $\nu_{\xi_m^i+1}(\mathsf{In}_i^\bullet) = \nu_{\xi_m^i}(\mathsf{In}_i^\bullet) \cup \{\lambda_m^i\}$, and so $n \in \nu_{\xi_m^i+1}(\mathsf{In}_i^\bullet)$. It follows that $n \in \nu_k(\mathsf{In}_i^\bullet)$ for every k with $\xi_m^i + 1 \le k$. As $\lambda_m^i > \xi_m^i$, we have $n \in \nu_n(\mathsf{C}_i^\bullet)$ as required.

Now it is not hard to check that

$$|\nu_n(\mathsf{In}_i^\circ) - \nu_n(\mathsf{Out}_i^\circ)| = |\nu_n(\mathsf{In}_i^\bullet) - \nu_n(\mathsf{Out}_i^\bullet)| = c_i(n)$$

and $\mathfrak{N}_\infty, \langle n, 0\rangle \models \mathsf{lin\_do}^\circ(\alpha_{n+1}) \wedge \mathsf{lin\_do}^\bullet(\alpha_{n+1})$, for all $i < N$ and $n < \omega$, and so $\mathfrak{N}_\infty, \langle 0,0\rangle \models \mathsf{lingrid} \wedge \psi_M$. ☐

Now Theorem 4 follows from Prop. 3, Lemmas 4.3 and 4.4.

In some cases, we can have stronger lower bounds than in Theorem 4. We call a frame $\langle W, R\rangle$ *modally discrete* if it satisfies the following aspect of discreteness: there are no points $x_0, x_1, \ldots, x_n, \ldots, x_\infty$ in W such that $x_0 R x_1 R x_2 R \ldots R x_n R \ldots R x_\infty$, $x_n \neq x_{n+1}$ and $x_\infty \neg R x_n$, for all $n < \omega$. We denote by **DisK4.3** the logic of all modally discrete linear orders. Several well-known 'linear' modal logics are extensions of **DisK4.3**, for example, $\mathsf{Logic\_of}\langle\omega, <\rangle$, $\mathsf{Logic\_of}\langle\omega, \leq\rangle$, **GL.3** (the unimodal logic of all Noetherian[3] linear orders), and **Grz.3** (the unimodal logic of all Noetherian reflexive linear orders). Unlike 'real' discreteness, modal discreteness can be captured by modal formulas, and each of the logics above is finitely axiomatisable [35, 6].

Theorem 5. *Let L_h be any Kripke complete logic such that L_h contains* **DisK4.3** *and $\langle\omega, <\rangle$ is a frame for L_h. Let L_v be any Kripke complete logic having an ω-fan among its frames. Then both $L_h \times^\delta L_v$ and $L_h \times^\delta_{sq} L_v$ are Π^1_1-hard.*

Proof. We sketch how to modify the proof of Theorem 4 to obtain a reduction of the 'CM recurrence' problem to $L_h \times^\delta L_v$-satisfiability. Observe that by Claim 4.1(ii),(iv), the generated grid-points x_n are such that $x_n \neq x_{n+1}$ for all $n < \omega$. Therefore, if $\mathfrak{M}$ is a model based on a δ-product frame with a modally discrete 'horizontal' component and

$$\mathfrak{M}, \langle r_h, r_v\rangle \models \mathsf{lingrid} \wedge \psi_M \wedge \Box_h \Diamond_h \Diamond_v (\delta \wedge \widehat{\mathsf{S}}_{q_r})$$

for some state q_r, then by Claim 4.1(iii),(iv), for every $n < \omega$ there is k such that $n < k < \omega$ and $\mathfrak{M}, \langle x_k, x_k\rangle \models \widehat{\mathsf{S}}_{q_r}$. □

However, the formula $\mathsf{lingrid}$ is clearly not satisfiable when L_h has only reflexive and/or dense frames (like **S4.3**, the unimodal logic of all reflexive linear orders, or the unimodal logic $\mathsf{Logic\_of}\langle\mathbb{Q}, <\rangle$ over the rationals). It is not hard to see that a 'linear' version of the 'tick-trick' in (18)–(19) can be used to generalise the proof of Theorem 4 for these cases. Further, as by Claim 4.1 the formula $\mathsf{lingrid}$ forces an infinite ascending chain of points, it is not satisfiable when L_h has only Noetherian frames (like **GL.3** or **Grz.3**). Similarly to the **K**-case in Section 5, it is also possible to generate an infinite grid and then emulate counter machine runs by going *backwards* in linear frames, and so to extend Theorem 4 to Noetherian cases. The interested reader should consult [17], where all these issues are addressed in detail.

7 Decidable δ-products

The following theorem shows that the unbounded width of the second-component frames is essential in obtaining the undecidability result of Theorem 2:

Theorem 6. *$L \times^\delta$ **Alt**(n) is decidable in* coNExpTime, *whenever L is* **K** *or* **Alt**(m), *for $0 < n, m < \omega$.*

Proof. We prove the theorem for $\mathbf{K} \times^\delta \mathbf{Alt}(n)$. The other cases are similar and left to the reader. We show (by selective filtration) that if some formula ϕ does not belong to $\mathbf{K} \times^\delta \mathbf{Alt}(n)$, then there exists a δ-product frame for $\mathbf{K} \times^\delta \mathbf{Alt}(n)$ whose size is exponential in ϕ where ϕ fails. It will also be clear that the presence or absence of the diagonal is irrelevant in our argument.

To begin with, we let $sub(\phi)$ denote the set of all subformulas of ϕ. For any $\psi \in sub(\phi)$, we denote by $hd(\psi)$ the maximal number of nested 'horizontal' modal operators ($\Diamond_h$ and

[3] $\langle W, R\rangle$ is *Noetherian* if it contains no infinite ascending chains $x_0 R x_1 R x_2 R \ldots$ where $x_i \neq x_{i+1}$ for $i < \omega$.

$\Box_h$) in ψ. Similarly, $vd(\psi)$ denotes the 'vertical' nesting depth of ψ. Now suppose that $\mathfrak{M}, \langle r_h, r_v\rangle \not\models \phi$ in some model $\mathfrak{M}$ that is based on the δ-product of $\mathfrak{F}_h = \langle W_h, R_h\rangle$ and some frame $\mathfrak{F}_v = \langle W_v, R_v\rangle$ for $\mathbf{Alt}(n)$. (Note that with δ in our language it is possible to force cycles in the component frames of a δ-product, so we cannot assume that $\mathfrak{F}_h$ and $\mathfrak{F}_v$ are trees.) For every $k \leq vd(\phi)$, we define

$$U_v^k := \{y \in W_v : \text{there is a } k\text{-long } R_v\text{-path from } r_v \text{ to } y\}.$$

The U_v^k are not necessarily disjoint sets for different k, but we always have

$$|U_v^k| \leq 1 + n + n^2 + \cdots + n^k \leq 1 + k \cdot n^k. \tag{28}$$

Then we define $\mathfrak{F}'_v := \langle W'_v, R'_v\rangle$ by taking

$$W'_v := \bigcup_{k \leq vd(\phi)} U_v^k, \qquad\qquad R'_v := R_v \cap (W'_v \times W'_v).$$

Next, for every $m \leq hd(\phi)$, we define inductively U_h^m and S_h^m as follows. We let $U_h^0 := \{r_h\}$ and $S_h^0 := \emptyset$. Now suppose inductively that we have defined U_h^m and S_h^m for some $m < hd(\phi)$. For all $x \in U_h^m$, $y \in W'_v$, and $\Diamond_h\psi \in sub(\phi)$ with $\mathfrak{M}, \langle x, y\rangle \models \Diamond_h\psi$, choose some $z_{x,y,\psi}$ from W_h such that $xR_h z_{x,y,\psi}$ and $\mathfrak{M}, \langle z_{x,y,\psi}, y\rangle \models \psi$. Then define

$$U_h^{m+1} := \{z_{x,y,\psi} : x \in U_h^m,\ y \in W'_v, \Diamond_h\psi \in sub(\phi),\ \mathfrak{M}, \langle x, y\rangle \models \Diamond_h\psi\},$$
$$S_h^{m+1} := \{\langle x, z_{x,y,\psi}\rangle : x \in U_h^m,\ y \in W'_v, \Diamond_h\psi \in sub(\phi),\ \mathfrak{M}, \langle x, y\rangle \models \Diamond_h\psi\}.$$

Again, the U_h^m are not necessarily disjoint sets for different m, but by (28) we always have that

$$|U_h^m| \leq \big(vd(\phi) \cdot n^{vd(\phi)} \cdot |sub(\phi)|\big)^m. \tag{29}$$

Then we define $\mathfrak{F}'_h := \langle W'_h, R'_h\rangle$ by taking

$$W'_h := \bigcup_{m \leq hd(\phi)} U_h^m, \qquad\qquad R'_h := \bigcup_{m \leq hd(\phi)} S_h^m.$$

Clearly, by (28) and (29) the size of $\mathfrak{F}'_h \times^\delta \mathfrak{F}'_v$ is exponential in the size of ϕ. Let $\mathfrak{M}'$ be the restriction of $\mathfrak{M}$ to $\mathfrak{F}'_h \times^\delta \mathfrak{F}'_v$. Now a straightforward induction on k, m and the structure of formulas shows that for all $k \leq vd(\phi)$, $m \leq hd(\phi)$, $\psi \in sub(\phi)$,

$$\mathfrak{M}, \langle x, y\rangle \models \psi \qquad \text{iff} \qquad \mathfrak{M}', \langle x, y\rangle \models \psi,$$

whenever $x \in U_h^{hd(\phi)-m}$, $y \in U_v^{vd(\phi)-k}$, $hd(\psi) \leq m$, and $vd(\psi) \leq k$. It follows that $\mathfrak{M}', \langle r_h, r_v\rangle \not\models \phi$, as required. $\Box$

In certain cases the above proof gives polynomial upper bounds on the size of the falsifying δ-product model, so we have:

Theorem 7. *The validity problems of both* $\mathbf{S5} \times^\delta \mathbf{Alt}(1)$ *and* $\mathbf{Alt}(1) \times^\delta \mathbf{Alt}(1)$ *are* coNP*-complete.*

Note that all the above results hold with $\mathbf{Alt}(n)$ being replaced by its *serial*[4] version $\mathbf{DAlt}(n)$. One should simply make the 'final' points in the filtrated component frames reflexive.

[4]A frame $\langle W, R\rangle$ is called *serial*, if for every x in W there is y with xRy.

8 Open problems

We have shown that in many cases adding a diagonal to product logics results in a dramatic increase in their computational complexity (Sections 5 and 6), while in other cases upper bounds similar to diagonal-free product logics can be obtained (Section 7). Here are some related open problems:

1. Theorems 4 and 5 do not apply when the first component logic has transitive but not necessarily weakly connected (linear) frames. In particular, while $\mathbf{K4} \times \mathbf{S5}$ is decidable in coN2ExpTime [8], it is not known whether $\mathbf{K4} \times^{\delta} \mathbf{S5}$ remains decidable. Note that it is not clear either whether we could somehow use Theorem 2 here, that is, whether $\mathbf{K} \times^{\delta} \mathbf{S5}$ could be reduced to $\mathbf{K4} \times^{\delta} \mathbf{S5}$. Note that the reduction of [13] from $\mathbf{K} \times L$ to $\mathbf{K4} \times L$ uses that $\mathbf{K} \times L$ is determined by product frames having intransitive trees as first components, and this is no longer true for $\mathbf{K} \times^{\delta} L$. As is shown in Lemma 2.4 and Claim 2.1, the formula grid defined in (6)–(7) is satisfiable in a δ-product frame for $\mathbf{K} \times^{\delta} L$, but forces a 'horizontal' non-tree structure.

2. By the above, $\mathbf{K} \times^{\delta} \mathbf{K}$ is properly contained in

   $$\mathsf{Logic\_of}(\text{'Intransitive trees'} \times^{\delta} \text{'Intransitive trees'}),$$

   and Theorem 2 does not imply the undecidability of the latter. Is this logic decidable? Note that it is not clear either whether the selective filtration proof of Theorem 6 could be used here, as both component frames could be of arbitrary width. However, it might be possible to generalise one of the several proofs showing the decidability of $\mathbf{K} \times \mathbf{K}$ [8, 7].

3. It can be proved using 2D type-structures called quasimodels that the diagonal-free product logic $\mathbf{K} \times \mathbf{Alt}(1)$ is decidable in ExpTime [7, Thm.6.6]. Is $\mathbf{K} \times^{\delta} \mathbf{Alt}(1)$ also decidable in ExpTime?

4. While δ-product logics are determined by δ-product frames by definition, there exist other (non-product, 'abstract') δ-frames for these logics. The *finite frame problem* of a logic L asks: "Given a finite frame, is it a frame for L?" If a logic L is finitely axiomatisable, then its finite frame problem is of course decidable: one just has to check whether the finitely many axioms hold in the finite frame in question. However, as is shown in [19], many δ-product logics ($\mathbf{K} \times^{\delta} \mathbf{K}$ and $\mathbf{K} \times^{\delta} \mathbf{K4}$ among them) are not finitely axiomatisable. So the decidability of the finite frame problem is open for these logics. Note that if every finite frame for, say, $\mathbf{K} \times^{\delta} \mathbf{K}$ were the p-morphic image of a finite δ-product frame, then we could enumerate finite frames for $\mathbf{K} \times^{\delta} \mathbf{K}$. As $\mathbf{K} \times^{\delta} \mathbf{K}$ is recursively enumerable by Theorem 1, we can always enumerate those finite δ-frames that are not frames for $\mathbf{K} \times^{\delta} \mathbf{K}$. So this would provide us with a decision algorithm for the finite frame problem of $\mathbf{K} \times^{\delta} \mathbf{K}$. However, consider the δ-frame $\mathfrak{F} = \langle W, R_h, R_v, D \rangle$, where

   $$\begin{aligned} W &= \{x, y, z\}, \qquad\qquad D = \{z\}, \\ R_h &= \{\langle x,x\rangle, \langle y,y\rangle, \langle z,z\rangle, \langle y,z\rangle, \langle z,x\rangle, \langle y,x\rangle\}, \\ R_v &= \{\langle x,x\rangle, \langle y,y\rangle, \langle z,z\rangle, \langle x,z\rangle, \langle z,y\rangle, \langle x,y\rangle\}. \end{aligned}$$

   Then it is easy to see that $\mathfrak{F}$ is a p-morphic image of $\langle \omega, \leq \rangle \times^{\delta} \langle \omega \leq \rangle$, but $\mathfrak{F}$ is not a p-morphic image of any finite δ-product frame.

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